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Functional Analysis - Exercises 7

Solutions

(1) $\mathcal{M} \subset \mathcal{H}$ closed subspace is invariant under $A \in B(\mathcal{H})$ iff $\mathcal{M}^\perp$ is invariant under A^*. Thus $\mathcal{M}$ reduces A iff it is invariant under both A and A^*.

Let $\mathcal{M} \subset \mathcal{H}$ be an invariant closed subspace under A ie

$$\forall x \in \mathcal{M} \Rightarrow Ax \in \mathcal{M} \iff \forall y \in \mathcal{M}^\perp : y \perp Ax \text{ ie}$$

$$\langle Ax, y\rangle = 0 = \langle x, A^* y\rangle \quad \forall x \in \mathcal{M}, y \in \mathcal{M}^\perp \iff$$

$A^* y \in \mathcal{M}^\perp$ for all $y \in \mathcal{M}^\perp$ ie $\mathcal{M}^\perp$ is invariant under A^*.

As $\mathcal{M}$ reduces A mean that ~~it is invariant but~~ $\mathcal{M}$ and $\mathcal{M}^\perp$ are invariant under A, the last statement is trivial.

(2) X, Y Banach spaces, $T \in B(X, Y)$ is bounded below ie $\exists\, C > 0$ s.t. $\|Tx\| \geq C\|x\| \ \forall x \in X$. $\Rightarrow$ Ran T is closed.

Let $(y_n)_{n\in\mathbb{N}} \subset \operatorname{Ran} T$ converging to $y \in Y$. We need to show that $y \in \operatorname{Ran} T$. There exists $(x_n)_{n\in\mathbb{N}} \subset X$ st $Tx_n = y_n$.

T is bounded below

$$\hookrightarrow \quad \|x_n - x_m\| \leq \frac{1}{C}\|Tx_n - Tx_m\| = \frac{1}{C}\|y_n - y_m\|$$

$\Rightarrow$ $(y_n)_{n\in\mathbb{N}}$ is Cauchy (it is convergent) $\Rightarrow$ $(x_n)_{n\in\mathbb{N}}$ is also Cauchy, hence convergent ~~because~~ because X is complete:

$\|x_n - x\| \to 0 \ (n\to\infty)$. Since T is continuous: $Tx_n = y_n \to Tx$

$\Rightarrow y = Tx \Rightarrow y \in \operatorname{Ran} T$.!

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(3) $T \in B(X)$

a) T is invertible $\Longleftrightarrow$ T is bounded below and $\operatorname{Ran} T$ is dense

b) $X = \mathcal{H}$: T is invertible $\Longleftrightarrow$ T is bounded below and $\ker T^* = \{0\}$

c) $X = \mathcal{H}$, T normal. T is invertible $\Longleftrightarrow$ T is bounded below

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a) Suppose that T is invertible $\Rightarrow$ T is surjective, so $\operatorname{Ran} T = Y$ is dense ✓

T^{-1} is bounded $\Rightarrow$ $\|x\| = \|T^{-1}Tx\| \leq \|T^{-1}\| \cdot \|Tx\|$

$\hookrightarrow$ $\|Tx\| \geq C \cdot \|x\|$ with $C = \dfrac{1}{\|T^{-1}\|}$ $\Rightarrow$ T is bounded below ✓

Suppose that T is bounded below and $\operatorname{Ran} T$ is dense.

By (2) $\operatorname{Ran} T$ is closed $\Rightarrow$ $\operatorname{Ran} T = X$ $\Rightarrow$ T is surjective.

T is injective, because if $Tx = 0$, then $\|x\| \leq \dfrac{\|Tx\|}{C} = 0$

$\Downarrow$

$x = 0$.

$\Rightarrow$ T is bijective and by the inverse mapping theorem it follows that T is invertible.

3/ b) $\operatorname{Ran} T$ is dense $\iff \overline{\operatorname{Ran} T} = \mathcal{H} \iff (\operatorname{Ker} T^*)^\perp = \mathcal{H} \iff$

$$\Uparrow$$

$$(\operatorname{Ker} T^*)^\perp = \overline{\operatorname{Ran} T}$$

$\iff \operatorname{Ker} T^* = \{0\}$

c) For a normal operator $\operatorname{Ker} T^* = \operatorname{Ker} T \implies T$ is invertible iff it is bounded below and $\operatorname{Ker} T = \{0\}$

$\Uparrow$ (b)

o!

4) X Banach space, $T \in B(X)$. Then $\operatorname{Ran} T$ is not dense iff $\exists\, \varphi \in X^*$, $\varphi \neq 0$ s.t. ~~$\varphi(T) = 0$~~ $\varphi \circ T = 0$

If there exists $\varphi \in X^*$ with $\varphi(\circ T) = 0 \implies \operatorname{Ran} T \subset \operatorname{Ker} \varphi$

However, $\operatorname{Ker} \varphi$ is a closed proper subspace of X, so the closure of $\operatorname{Ran} T$ is also contained by $\operatorname{Ker} \varphi$: $\overline{\operatorname{Ran} T} \subset \operatorname{Ker} \varphi \neq X$

$$\Downarrow$$

$\operatorname{Ran} T$ is not dense

✓

For the converse, suppose that $\operatorname{Ran} T$ is not dense

$$\Downarrow$$

$Y := \overline{\operatorname{Ran} T}$ is a closed proper subspace of X.

For any $x \in X \setminus Y$, by the Hahn-Banach Thm $\exists\, \varphi \in X^*$ s.t. $\varphi(y) = 0$ for any $y \in Y$ and $\varphi(x) = 1$

$$\Downarrow$$

$\varphi \neq 0$ and $\varphi \circ T = 0$

o!

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(5) $\mathcal{H}$ Hilbert space, $T \in \mathcal{B}(\mathcal{H})$ normal. Then

a) $\operatorname{Ker} T$ is T^*-invariant.

b) $(\operatorname{Ker} T)^\perp$ is T-invariant

c) $\operatorname{Ker} T = \operatorname{Ker} T^2$

d) $\operatorname{Ker} T = \operatorname{Ker} T^k \quad k \in \mathbb{Z}^+$

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a) $x \in \operatorname{Ker} T \Rightarrow Tx = 0$

$$\Rightarrow TT^*x \underset{\substack{\uparrow \\ T \text{ is normal}}}{=} T^*Tx \overset{\Downarrow}{=} T^*0 = 0 \Rightarrow T^*x \in \operatorname{Ker} T \checkmark$$

b) $Tx \in (\operatorname{Ker} T)^\perp \ \forall x \in \mathcal{H}$: for every $y \in \operatorname{Ker} T = \operatorname{Ker} T^*$

$$\langle Tx, y \rangle = \langle x, T^*y \rangle = \langle x, 0 \rangle = 0$$

$$\Downarrow$$

$$Tx \perp y \rightsquigarrow Tx \in (\operatorname{Ker} T)^\perp$$

d) (c is special case)

$\forall x \in \mathcal{H} \quad \exists y \in \operatorname{Ker} T$ and $z \in (\operatorname{Ker} T)^\perp$ s.t. $x = y + z$.

$$\Rightarrow Tx = Ty + Tz = Tz \Rightarrow T^k \cancel{x} = T^k z \quad \forall k \in \mathbb{Z}^+$$

b) $\Rightarrow$ $(\operatorname{Ker} T)^\perp$ is T invariant and $\operatorname{Ker} T \cap (\operatorname{Ker} T)^\perp = \{0\}$,

so if we restrict T to $(\operatorname{Ker} T)^\perp$ then the operator

$(\operatorname{Ker} T)^\perp \to (\operatorname{Ker} T)^\perp$ is injective

$$\Rightarrow z = 0 \Leftrightarrow Tz = 0 \Leftrightarrow T^2 z = 0 \Leftrightarrow \ldots$$

!

J/

(6) $T \in \mathcal{B}(\mathcal{H})$ selfadjoint. If $\langle Tx, x\rangle = 0 \ \forall x \in \mathcal{H} \Rightarrow T = 0$.

We have seen in the lecture that for any selfadjoint operator

$$\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle|$$

$\Rightarrow$ If $\langle Tx, x\rangle = 0 \ \forall x \in \mathcal{H} \Rightarrow \|T\| = 0 \Rightarrow T = 0$ o.!

(7) $T \in \mathcal{B}(\mathcal{H})$ is normal iff $\|Tx\| = \|T^*x\| \quad \forall x \in \mathcal{H}$.

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle$$

$$\|T^*x\|^2 = \langle T^*x, T^*x\rangle = \langle TT^*x, x\rangle$$

Since T^*T and TT^* are selfadjoint, we can use (6), and

$$\|Tx\| = \|T^*x\| \ \forall x \in \mathcal{H} \iff \langle (T^*T - TT^*)x, x\rangle = 0 \ \forall x \in \mathcal{H}$$

$$\Updownarrow$$

$$T^*T - TT^* = 0$$

$$\Updownarrow$$

$$T^*T = TT^*$$

o.!

(8) $T = \ell_2 \to \ell_2$, $T(x_1, x_2, \dots) = \left(\frac{x_1+x_2}{2}, \frac{x_2+x_3}{2}, \dots\right)$, $T^* = ?$

With the left shift operator $S_L : (x_1, x_2, \dots) \mapsto (x_2, x_3, \dots)$

$T = \frac{1}{2}(I + S_L)$. Since $\underset{\text{the right shift}}{S_L^* = S_R} \Rightarrow T^* = \frac{1}{2}(I + S_R)$ ie

$$T^*(x_1, x_2, \dots) = \left(\frac{x_1}{2}, \frac{x_1+x_2}{2}, \frac{x_2+x_3}{2}, \dots\right)$$

o.!

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(5) $T: \ell_\infty \to \ell_\infty$, $T(x_1, x_2, x_3, \dots) = \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \frac{x_4}{4} \dots\right)$

Show that T is compact. Determine its spectrum.

Consider the operators:

$$T_n : (x_1, x_2, x_3, \dots) \longmapsto \left(x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots, \frac{x_n}{n}, 0, 0 \dots\right)$$

it is a finite rank operator for all $n \in \mathbb{N}$.

$$\|T - T_n\| = \frac{1}{n+1} \to 0 \quad \text{i.e.} \quad T_n \to T, \infty$$

$$(T - T_n)x = \left(0, \dots, 0, \frac{x_{n+1}}{n+1}, \frac{x_{n+2}}{n+2}, \dots\right)$$

T is a limit of finite rank operators

$\Downarrow$

T is compact ✓

- For the point spectrum: $Tx = \lambda x$

$\Updownarrow$

$x = (0, \dots, \underset{n}{\cancel{1}}, 0, \dots)$ and $\lambda = \frac{1}{n}$ $\forall n \in \mathbb{N}$

$\Rightarrow \boxed{\sigma_p = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}}$

For a compact operator the spectrum is the same as the point spectrum except maybe the point 0.

The image of a bounded sequence $x = (x_1, x_2 \dots)$ is a sequence converging to 0 $\Rightarrow \operatorname{Ran} T \subset c_0$

c_0: ~~the~~ sequences converging to zero

Since c_0 is closed in ℓ_∞ $\Rightarrow \overline{\operatorname{Ran} T} \subset c_0 \Rightarrow \operatorname{Ran} T$ is not dense

$\Rightarrow 0 \in \sigma_r(T)$.

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So the spectrum of T: $\sigma_p(T) = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$ the point spectrum

$\sigma_r(T) = \{0\}$ the residual spectrum

$\sigma_c(T) \cancel{\neq} \phi$ continuous spectrum

(10) $T: (x_1, x_2, \dots) \mapsto \left(x_1, \frac{x_1+x_2}{2}, \frac{x_1+x_2+x_3}{3}, \dots\right)$

$\ell_\infty \to \ell_\infty$ operator

- T is bounded, $\|T\| = ?$
- Is T injective, surjective?

For $x = (x_1, x_2, \dots)$

$$\left| \frac{x_1 + x_2 + \dots + x_n}{n} \right| \le \frac{|x_1| + \dots + |x_n|}{n} \le \frac{n \cdot \|x\|_\infty}{n} = \|x\|_\infty$$

$\Rightarrow \|Tx\|_\infty \le \|x\|_\infty \Rightarrow \|T\| \le 1$

For $x = (1, 1, \dots)$, $Tx = x \Rightarrow \boxed{\|T\| = 1}$

- Injectivity:

Suppose $Tx = 0 \Rightarrow x_1 = 0$

$\hookrightarrow \frac{x_1 + x_2}{2} = \frac{x_2}{2} = 0 \Rightarrow x_2 = 0$

$\frac{x_1 + x_2 + x_3}{3} = \frac{x_3}{3} = 0 \Rightarrow x_3 = 0, \dots$

$\Rightarrow x = 0 \Rightarrow$ T is injective

- Surjectivity

For $y = (1, 0, 1, 0, 1, \dots) \in \ell_\infty$ we look for $x \in \ell_\infty$ s.t. $Tx = y$.

$$\Longrightarrow \quad \begin{array}{l} x_1 = 1 \\ x_2 = -1 \\ x_3 = 3 \\ x_4 = -3 \\ x_5 = 5 \\ x_6 = -5 \\ \vdots \end{array} \quad \Longrightarrow \quad \begin{array}{l} x_{2k+1} = 2k+1 \\ x_{2k+2} = -(2k+1) \end{array}$$

$$\Downarrow$$

$x \notin l_\infty$ so, $y \notin \operatorname{Ran} T$

$$\Downarrow$$

T is not surjective !

(11) Volterra operator on $L^2[0,1]$:

$$(Vf)(x) = \int_0^x f(t)\,dt.$$

a) V is compact

b) $V^* = ?$, it is selfadjoint?

c) Its spectrum is $\{0\}$, it has no eigenvalues

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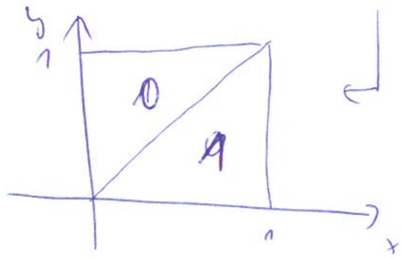
a) V can be given as an integral operator:

$$(Vf)(x) = \int_0^x f(t)\,dt = \int_0^1 K(x,y)\,f(y)\,dy$$

with

$$K(x,y) = \begin{cases} 1, & \text{if } 0 \le y < x \\ 0, & \text{if } x < y \le 1 \end{cases}$$

and $\displaystyle\int_0^1\int_0^1 |K(x,y)|^2\,dx\,dy < \infty$



$$\Downarrow$$

V is an integral operator

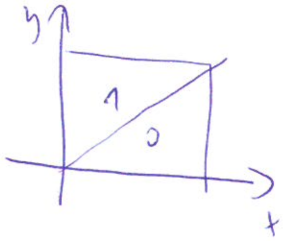
$$\Downarrow$$

compact.

5/ b)

$\Rightarrow$ The adjoint of V is an integral operator with a kernel

$$K^*(x,y) = \overline{K(y,x)} = \begin{cases} 1 & \text{if } 0 \le x < y \\ 0 & \text{if } y < x \le 1 \end{cases}$$



$$\Rightarrow (V^* f)(x) = \int_0^1 K^*(x,y) f(y)\,dy = \int_x^1 f(y)\,dy$$

$\Rightarrow$ $V \neq V^*$ $\rightsquigarrow$ V is not selfadjoint

c) To see that the spectrum is at most $\{0\}$, we show that the spectral radius is 0:

$$(V^n f)(x) = \int_0^x \int_0^{x_{n-1}} \cdots \int_0^{x_2} \int_0^{x_1} f(t)\,dt\,dx_1\,dx_2 \ldots dx_{n-1} =$$

$$= \int_0^x f(t) \left(\int_t^x \int_t^{x_{n-1}} \cdots \int_t^{x_2} dx_1\,dx_2 \ldots dx_{n-1} \right) dt =$$

$0 \le t \le x_1$
$0 \le x_1 \le x_2$
$\vdots$

$$= \int_0^x f(t) \frac{(x-t)^{n-1}}{(n-1)!}\,dt$$

$$\Rightarrow \quad \|V^n\| \le \frac{1}{n!}$$

We need to compute $\lim_{n\to\infty} \|V^n\|^{1/n}$.

10)

$$\log \lim_{2n\to\infty} \left(\frac{1}{(2n)!}\right)^{1/2n} = -\lim_{2n\to\infty} \frac{1}{2n} \log (2n)! =$$

$$= -\lim \frac{1}{2n} \underbrace{\sum_{1\le k\le 2n} \log k}_{\sum_{1\le k\le n} (\log k + \log(2n-k+1))} \le$$

$$\le -\lim \frac{1}{2n} \sum_{1\le k\le \frac{n}{4}} (\log k + \log(2n-k+1)) \le$$

$$\le -\lim \frac{1}{2n} \sum_{1\le k\le \frac{n}{4}} \log 2n = -\lim \frac{\log 2n}{2} = -\infty$$

$$\Downarrow$$

the spectral radius: $\lim_{n\to\infty} \|V^n\|^{1/n} = 0$

$\Downarrow$ the spectrum is not empty

$$\boxed{\sigma(V) = \{0\}}$$

For $f \in L^2[0,1]$ $Vf = 0 \Rightarrow Vf$ is almost everywhere 0

Since $x \mapsto (Vf)(x)$ is continuous $\Rightarrow (Vf)(x) = 0 \ \forall x \in [0,1]$

$$\Downarrow$$

$$f = 0$$

$$\Downarrow$$

there are no eigenvectors of V !

(12) Let $T \in \mathcal{B}(\mathcal{H})$ arbitrary. Prove that

a) $\sigma(T^*) = \{\bar{\lambda} : \lambda \in \sigma(T)\}$

b) $\lambda \in \sigma_p(T)$ iff $\mathrm{Ran}(\bar{\lambda} I - T)$ is not dense.

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a) $\lambda \in \sigma(T)$ iff $\lambda I - T$ is not invertible

If S is invertible, then S^* is invertible and since $(S^*)^* = S$, we have

$$\lambda \in \sigma(T) \iff (\lambda I - T) \text{ is not invertible} \iff$$

$$\iff (\lambda I - T)^* = (\bar{\lambda} I - T^*) \text{ is not invertible} \iff \bar{\lambda} \in \sigma(T^*).$$

b) Recall: $(\mathrm{Ker}\, S)^\perp = \overline{\mathrm{Ran}\, S^*}$

For $S = \lambda I - T$:

$$\lambda \in \sigma_p(T) \iff \mathrm{Ker}(\lambda I - T) \neq \{0\} \iff \overline{\mathrm{Ran}(\lambda I - T)^*} \neq \mathcal{H} \iff$$

$$\iff \mathrm{Ran}(\bar{\lambda} I - T^*) \text{ is not dense.}$$

o!

(13) The residual spectrum of a normal operator $T \in \mathcal{B}(\mathcal{H})$ is empty.

$\sigma_r(T) = \{\lambda : \mathrm{Ker}(\lambda I - T) = \{0\} \text{ and } \mathrm{Ran}(\lambda I - T) \text{ is not dense}\}$

If T is normal, then $\mathrm{Ker}\, T = \mathrm{Ker}\, T^* = (\mathrm{Ran}\, T)^\perp$

↳ For a normal operator T $\mathrm{Ker}\, T = \{0\} \iff \mathrm{Ran}\, T$ is dense

Since $\lambda I - T$ is normal $\implies$ $\sigma_r(T) = \emptyset$

o!

12/

(17) Let $T \in \mathcal{B}(\mathcal{H})$ be normal. Then

a) $\|T^2\| = \|T\|^2$

b) $\|T^{2^k}\| = \|T\|^{2^k}$ $\quad k \in \mathbb{Z}^+$

c) $r(T) = \|T\|$

d) $\|T^n\| = \|T\|^n$ $\quad \forall n \in \mathbb{Z}^+$

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a) Since $(T^2)^* = (TT)^* = T^*T^* = (T^*)^2$ and $(T^*T)^* = T^*T$

we use the equality $\|S^*S\| = \|S\|^2$ for $S = T^2$:

$$\underbrace{\|T^*T^*TT\|}_{\sim\sim} = \|T^2\|^2$$

and for $S = T^*T$: $\underbrace{\|T^*TT^*T\|}_{---} = \|T^*T\|^2 = \|T\|^4$

For a normal operator $\sim\sim$ and $---$ coincide

$\Rightarrow$ $\|T^2\| = \|T^*T\| = \|T\|^2$.

b) T is normal $\Rightarrow$ T^2 is normal $\Rightarrow$ we can use a) for $T, T^2, \dots, T^{2^{k-1}}$:

$$\|T^{2^k}\| = \|T^{2^{k-1}}\|^2 = \|T^{2^{k-2}}\|^4 = \dots = \|T\|^{2^k}.$$

c) since $r(T) = \lim_{n\to\infty} \sqrt[n]{\|T^n\|}$, for $n = 2^k$ we have $\sqrt[n]{\|T^n\|} \underset{(b)}{=} \|T\|$

$\Rightarrow$ $r(T) = \|T\|$

d) since $r(T) = \inf_n \sqrt[n]{\|T^n\|}$

and $r(T) = \|T\| \Rightarrow \|T^n\| \geq r(T)^n = \|T\|^n$ and $\|T^n\| \leq \|T\|^n$ for any op. !