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Functional Analysis – Exercises 7

Solutions II.

(15) What is the set of eigenvalues for the left shift operator

$$S_L : (x_1, x_2, x_3, \dots) \mapsto (x_2, x_3, x_4, \dots)$$

a) as an $\mathbb{C}^{\mathbb{N}} \to \mathbb{C}^{\mathbb{N}}$ operator

b) $l_\infty \to l_\infty$ operator

c) $l_p \to l_p$ operator

If $S_L x = \lambda x$ for some nonzero $x = (x_1, x_2, \dots)$, then we have

$$(x_2, x_3, x_4, \dots) = (\lambda x_1, \lambda x_2, \lambda x_3, \dots)$$

with the choice $x_1 = 1 \implies x_2 = \lambda,\ x_3 = \lambda^2, \dots$

So we need to determine the set of those λ for which the vector $(1, \lambda, \lambda^2, \dots)$ is in the spaces $\mathbb{C}^{\mathbb{N}}, l_\infty, l_p$

a) $(1, \lambda, \lambda^2, \lambda^3, \dots) \in \mathbb{C}^n$ for all $\lambda \in \mathbb{C} \implies \boxed{\sigma_p(S_L) = \mathbb{C}}$

b) $(1, \lambda, \lambda^2, \dots) \in l_\infty \iff \sup_\lambda |\lambda^n| < \infty\ \forall n \iff |\lambda| \leq 1$

$$\Downarrow$$

$$\boxed{\sigma_p(S_L) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}}$$

c) $(\cancel{1}, \lambda, \lambda^2, \dots) \in l_p \iff \sum_{k=0}^{\infty} |\lambda^k|^p = \sum_{k=0}^{\infty} (|\lambda|^p)^k < \infty$

$\forall 1 \leq p < \infty$

$$\Downarrow$$

$|\lambda| < 1 \implies \boxed{\sigma_p(S_L) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}}$

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(16) Consider the left shift S_L and the right shift

$S_R : (x_1, x_2, \dots) \longmapsto (0, x_1, x_2, \dots)$ as $\ell_2 \to \ell_2$ operators!

Determine their adjoints and find their point, cont., and residual spectrum. What about, when we consider them as $\ell_2(\mathbb{Z}) \to \ell_2(\mathbb{Z})$ operators?

- the adjoint of S_L

for $x = (x_1, x_2, x_3, \dots) \in \ell_2$
$y = (y_1, y_2, \dots) \in \ell_2$

$$\langle S_L x, y \rangle = \langle (x_2, x_3, x_4, \dots), (y_1, y_2, y_3, \dots) \rangle =$$

$$= x_2 \overline{y_1} + x_3 \overline{y_2} + x_4 \overline{y_3} + \dots = x_1 \cdot 0 + x_2 \cdot \overline{y_1} + x_3 \overline{y_2} + \dots =$$

$$= \langle (x_1, x_2, x_3, \dots), (0, y_1, y_2, \dots) \rangle = \langle x, S_R y \rangle$$

$$\Rightarrow \boxed{S_L^* = S_R} \Rightarrow (S_L^*)^* = S_L = S_R^*$$

$$\Downarrow$$

$$\boxed{S_R^* = S_L}$$

- We have already seen that $\boxed{\|S_L\| = \|S_R\| = 1}$

$$\Downarrow$$

$\{\lambda \in \mathbb{C} \mid |\lambda| > 1\}$ is contained in the resolvent sets of both S_L and S_R.

- We have seen in (15) that the point spectrum of S_L is

$$\boxed{\sigma_p(S_L) = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\}}$$

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- <u>the point spectrum of S_R:</u>

$$S_R(x_1, x_2, \dots) = \lambda(x_1, x_2, x_3, \dots)$$

$$\Longleftrightarrow \quad (0, x_1, x_2, \dots) = (\lambda x_1, \lambda x_2, \lambda x_3, \dots)$$

$$\Longleftrightarrow \quad (x_1, x_2, x_3, \dots) = (0, 0, 0, \dots)$$

$$\Longrightarrow \quad \boxed{\sigma_p(S_R) = \emptyset}$$

- <u>Other spectrum of S_L:</u>

Since the spectrum of any operator is closed and since $\|S_L\| = 1$, we must have for the resolvent set

$$\rho(S_L) = \{\lambda \in \mathbb{C} : |\lambda| > 1\}$$ and for

the spectrum $\sigma(S_L) = \{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$

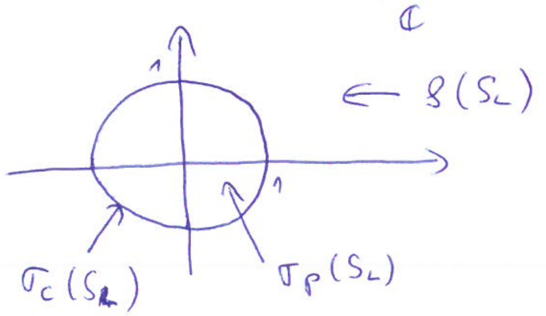
As $\sigma_p(S_L) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$, the only remaining question is what part of the unit circle $|\lambda| = 1$ consist of residual spectrum.

We proved in the last lecture:

$$\boxed{\text{If } T \in \mathcal{B}(\mathcal{H}),\ \lambda \in \sigma_r(T) \Rightarrow \bar{\lambda} \in \sigma_p(T^*)}$$

↳ If λ were in the residual spectrum of S_L, $\bar{\lambda}$ would be in the point spectrum of $S_L^* = S_R$ and we know S_R has no point spectrum, so $\boxed{\sigma_r(S_L) = \emptyset}$

$\Longrightarrow$



An other proof was given in the lecture !

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• other spectrum of S_R:

We have seen in the lecture:

$$\boxed{T \in \mathcal{B}(\mathcal{H}) \quad \lambda \in \sigma_p(T) \Longrightarrow \bar{\lambda} \in \sigma_p(T^*) \cup \sigma_r(T^*)}$$

$$\Rightarrow \text{ If } |\lambda| < 1 \text{, then } \lambda \in \sigma_p(S_L) \xRightarrow[S_L^* = S_R]{} \bar{\lambda} \in \sigma_p(S_R) \cup \sigma_r(S_R)$$

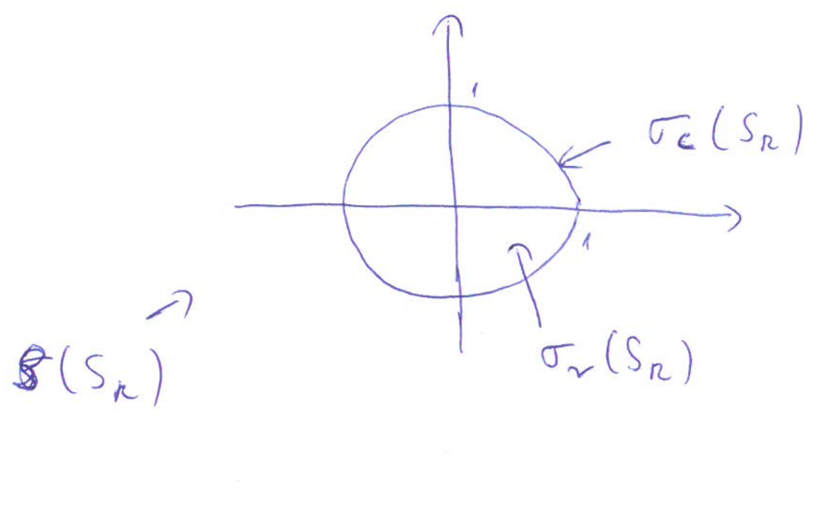
as $\sigma_p(S_R) = \emptyset \Rightarrow \{\lambda \in \mathbb{C} : |\lambda| < 1\} \subset \sigma_r(S_R)$.

Since the spectrum of any operator is closed and since $\|S_R\| = 1$

$$\Rightarrow \rho(S_R) = \{\lambda \in \mathbb{C} \mid |\lambda| > 1\} \text{ and } \sigma(S_R) = \{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}.$$

So the only remaining question is which part of the unit circle $|\lambda| = 1$ consist of residual spectrum.

But if some λ with $|\lambda| = 1$ were in the residual spectrum of S_R, $\bar{\lambda}$ would be in the point spectrum of $S_R^* = S_L$ and we know that the point spectrum of S_L does not intersect the unit circle: so $\sigma_r(S_R) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$.



5/ • If we consider S_R and S_L as operators on $\ell_2(\mathbb{Z})$, then for any

$$x = (\dots, x_{-1}, x_0, x_1, \dots) \in \ell_2(\mathbb{Z})$$

$$(S_R x)_n = x_{n-1} \quad \text{and} \quad (S_L x)_n = x_{n+1}.$$

It easy to check similarly to the case of $\ell_2(\mathbb{N})$, that S_L and S_R are mutual adjoints : $S_L^* = S_R$ and $S_R^* = S_L$ and mutual inverses : $S_L^{-1} = S_R$

$$\Downarrow$$

<u>they are unitaries</u>

Being unitary, their operator norms are 1, so their spectra are non-empty compact subset of the unit ~~circle~~. circle.

• <u>point spectrum:</u>

for $S_R x = \lambda x$, if there is any index n with $x_n \neq 0$, then $S_R x = \lambda x$ gives $x_{n+k+1} = \lambda x_{n+k}$ for $k = 0, 1, 2, \dots$

Since $|\lambda| = 1$, such a vector is not in $\ell_2(\mathbb{Z})$

$$\Downarrow$$

<u>they have no eigenvalues !</u>

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(17) X Banach space, $A \in \mathcal{B}(X)$, $\lambda \in \mathbb{C}$. Assume $\exists\ (x_n)_{n\in\mathbb{N}} \subset X$ s.t. $\|x_n\| = 1$ and $Ax_n - \lambda x_n \to 0$ as $n \to \infty$.

$\Longrightarrow \lambda \in \sigma(A)$.

If towards a contradiction we assume, that $\lambda \in \rho(A)$, then $A - \lambda I$ is a bijective bounded operator with bounded inverse. This implies:

$$1 = \|x_n\| = \|(A-\lambda I)^{-1}(A-\lambda I)x_n\| \le \|(A-\lambda I)^{-1}\| \cdot \|Ax_n - \lambda x_n\| \xrightarrow[n\to\infty]{} 0$$

$$\lightning \quad \Longrightarrow \quad \lambda \in \sigma(A)$$

o!

(18) $T \in \mathcal{B}(\mathcal{H})$ selfadjoint, $\alpha \in \mathbb{C}$, $\operatorname{Im} \alpha \neq 0$. $\Longrightarrow$ $U = (\bar{\alpha} I + T)(\alpha + T)^{-1}$ is unitary.

$T \in \mathcal{B}(\mathcal{H})$ is selfadjoint $\Longrightarrow \sigma(T) \subset \mathbb{R} \Longrightarrow \alpha I - T$ is invertible if $\operatorname{Im} \alpha \neq 0$.

Similarly for $(-\alpha)$ $\Longrightarrow$ $-\alpha I - T$ is invertible $\Longrightarrow$ $\alpha I + T$ is invertible

- for any $\beta \in \mathbb{C}$ $(\alpha I + T)^{-1}$ and $(\beta - \alpha) I$ commutes

$+$ $(\alpha I + T)^{-1}$ and $(\alpha I + T)$ commute

$\Downarrow$

$(\alpha I + T)^{-1}$ commutes with $(\beta - \alpha) I + (\alpha I + T) = \beta I + T$

$\Downarrow$

$\bar{\alpha} I + T$ and $(\alpha I + T)^{-1}$ commute

7) We need to show that $UU^* = U^*U = I$. Since $T^* = T$

$\hookrightarrow$ $UU^* = \underbrace{(\bar{\alpha} I + T)(\alpha I + T)^{-1}}_{\text{they commute}} \left[((\alpha I + T)^{-1})^* (\bar{\alpha} I + T)^* \right] =$

$= (\alpha I + T)^{-1} (\bar{\alpha} I + T) \left[((\alpha I + T)^*)^{-1} (\alpha I + T) \right]$

$= (\alpha I + T)^{-1} \underbrace{(\bar{\alpha} I + T)(\bar{\alpha} I + T)^{-1}}_{I} (\alpha I + T) = (\alpha I + T)^{-1}(\alpha I + T) = I$ ✓

$U^*U = I$ is similar.

∘!

(19) $T \in B(\mathcal{H}) \Rightarrow \operatorname{Ker}(T^*T) = \operatorname{Ker} T$.

If $Tx = 0 \Rightarrow STx = 0 \Rightarrow \operatorname{Ker} T \subset \operatorname{Ker}(ST)$ for any $S, T \in B(\mathcal{H})$

$\Rightarrow \operatorname{Ker} T \subset \operatorname{Ker} T^*T$ ✓

It remains to show $\operatorname{Ker} T^*T \subset \operatorname{Ker} T$.

Let $x \in \operatorname{Ker} T^*T$ i.e. $T^*Tx = 0$

$\hookrightarrow$ $\|Tx\|^2 = \langle Tx, Tx \rangle = \langle T^*Tx, x \rangle = \langle 0, x \rangle = 0 \Rightarrow Tx = 0$

$\Downarrow$

$x \in \operatorname{Ker} T$ ∘!

(20) $T, S \in B(\mathcal{H})$. a) $\lambda, \mu \in \rho(T)$

$$R_\lambda(T) - R_\mu(T) = (\lambda - \mu) R_\lambda(T) R_\mu(T)$$

b) $\lambda \in \rho(T) \cap \rho(S)$

$$R_\lambda(T) - R_\lambda(S) = R_\lambda(T)(T - S) R_\lambda(S)$$

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$$R_\lambda(A) - R_\mu(A) = (\lambda I - A)^{-1} - (\mu I - A)^{-1} =$$

$$= (\lambda I - A)^{-1} [\underbrace{\mu I - A - (\lambda I - A)}_{(\mu-\lambda) I}] (\mu I - A)^{-1} =$$

$$= (\mu - \lambda) R_\lambda(A) R_\mu(A)$$

b) $$R_\lambda(T) - R_\lambda(S) = (\lambda I - T)^{-1} - (\lambda I - S)^{-1} =$$

$$= (\lambda I - T)^{-1} [\underbrace{\lambda I - S - (\lambda I - T)}_{T-S}] (\lambda I - S)^{-1} =$$

$$= R_\lambda(T) [T-S] R_\lambda(S)$$

.!

21 $T: C[0,1] \to C[0,1]$, $(Tf)(x) = x f(x)$. Determine the spectrum.

$$[(\lambda I - T) f](x) = (\lambda - x) f(x)$$

• $\lambda I - T$ is injective for any $\lambda \in \mathbb{R}$:

Suppose $f \in \ker(\lambda I - T)$. Then $(\lambda - x) f(x) = 0 \quad \forall x \in [0,1]$ ie.

$f(x) = 0$ if $x \in [0,1] \setminus \{\lambda\}$. Since f is continuous $\Rightarrow f(x) = 0 \ \forall x \in [0,1]$ ✓

$\Downarrow$

$$\boxed{\sigma_p(T) = \emptyset}$$

• If $\lambda \notin [0,1]$, then $\lambda I - T$ is surjective:

for $g \in C[0,1]$, let $f(x) = \dfrac{g(x)}{\lambda - x}$ $\rightsquigarrow$ f is continuous and $(\lambda I - T) f = g$ ✓

• If $\lambda \in [0,1]$, then $\mathrm{Ran}(\lambda I - T)$ is not dense: $\forall f \in C[0,1]$ we have

$[(\lambda I - T) f](\lambda) = 0 \Rightarrow \mathrm{Ran}(\lambda I - T) \subset \{g \in C[0,1] : g(\lambda) = 0\} \leftarrow$ closed proper subspace of $C[0,1]$

$$\Rightarrow \boxed{\sigma_r(T) = \sigma(T) = [0,1]}$$

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