The Spectrum of an operator

Recall from Linear Algebra

In finite dimensional Hilbert spaces ie in $\mathcal{H} = \mathbb{C}^n$ every $A \in B(\mathcal{H})$ can be represented by an $n \times n$ complex matrix:

$\lambda \in \mathbb{C}$ called an eigenvalue of A corresponding to the eigenvector $v \in \mathcal{H} \equiv \mathbb{C}^n$ with $\underline{v} \neq 0$ if

$$Av = \lambda v$$

Fact: Every operator on an n-dimensional space has at least one and at most n eigenvalues.

It is not further true in infinite dimension.

Ex| Let S_r be the right shift op. on l_p, $1 \leq p \leq \infty$ ~~ie~~. For any $\lambda \in \mathbb{C}$ the equation $S_r x = \lambda x$, ie

$$(0, x_1, x_2, \dots) = \lambda (x_1, x_2, \dots)$$ yields the equations

$$\begin{aligned} 0 &= \lambda x_1 \\ x_1 &= \lambda x_2 \\ x_2 &= \lambda x_3 \\ &\vdots \end{aligned}$$

$\Longrightarrow$ can never satisfied by any nonzero vector x.

$\Downarrow$

S does not have any ~~eg~~ eigenvalue.

In finite dimension :

$$Av = \lambda v \qquad v \neq 0$$

$$\Longleftrightarrow (A - \lambda I)v = 0 \Longleftrightarrow \underset{\neq 0}{v} \in \ker(A - \lambda I)$$

$$\Longleftrightarrow \ker(A - \lambda I) \neq \{0\} \Longleftrightarrow A - \lambda I \text{ is } \underline{\text{not invertible}}$$

$$\left(\det(A - \lambda I) = 0 \right.$$
$$\left. \text{characteristic polynomial} \right)$$

<u>In finite dimension</u> :

$$A \text{ is invertible} \Longleftrightarrow \underset{A \text{ is injective}}{\ker A = \{0\}} \Longleftrightarrow \underset{A \text{ is surjective}}{\operatorname{Ran} A = \mathbb{C}^n} .$$

In infinite dimensional spaces it is different :

<u>Example</u> (1) Let T is given by its matrix $\begin{pmatrix} 1/1 & & 0 \\ & 1/2 & \\ 0 & & 1/3 \\ & & & \ddots \end{pmatrix}$.

$$\Rightarrow \ker T = \{0\} , T \text{ is injective}$$

but T is <u>not surjective</u>, for example the sequence

$$y = (1, 1/2, 1/3, \ldots) \in \ell^2 \text{ but } \notin \operatorname{Ran} T .$$

$$(\; Tx = y \Longleftrightarrow x = (1, 1, \ldots) \notin \ell^2 \;).$$

Recall: (Neumann-series of Banach operators)

Let X be a Banach space, $A \in \mathcal{B}(X)$, $\lambda \in \mathbb{C}$.

If $\|A\| < |\lambda|$, then the series

$$\frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{1}{\lambda^n} A^n \quad \text{absolutely convergent in } \mathcal{B}(X)$$

and its limit is $(\lambda I - A)^{-1}$, i.e.

$$(\lambda I - A)^{-1} = \frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{1}{\lambda^n} A^n .$$

Corollary: If $T \in \mathcal{B}(X)$ and $\|I - T\| < 1$, then T is invertible.

Proof. $\|\underbrace{I-T}_{A}\| < \underset{\lambda}{\underset{\|}{1}}$ $\qquad [I - (I-T)]^{-1} = T^{-1} = \sum_{n=0}^{\infty} (I-T)^n$

o!

Def: Let $T \in \mathcal{B}(\mathcal{H})$. The resolvent set of T is the collection of all complex numbers λ for which the operator $\lambda I - T$ ~~has~~ is invertible, i.e. it has a bounded inverse $(\lambda I - T)^{-1} \in \mathcal{B}(\mathcal{H})$.

Remark. The boundedness of the inverse is automatic from the inverse mapping theorem.

② The left shift (backward) S_ℓ is <u>surjective</u>:

given $(y_1, y_2, \dots) \in \ell^2$ $\quad S_\ell(0, y_1, y_2, \dots) = (y_1, y_2, \dots)$.

On the other hand $\quad S_\ell e_1 = S_\ell(1, 0, 0, \dots) = (0, 0, \dots)$

$$\Downarrow$$

S_ℓ is <u>not injective</u>

Also, $\quad S_\ell S_r(x_1, x_2, \dots) = S_\ell(0, x_1, x_2, \dots) = (x_1, x_2, \dots)$

$$\Rightarrow \quad S_\ell S_r = I$$

but $\quad S_r S_\ell(x_1, x_2, \dots) = S_r(x_2, x_3, \dots) = (0, x_2, x_3, \dots) \neq$

$\neq (x_1, x_2, \dots)$

$$\Rightarrow \quad S_r S_\ell \neq I$$

i.e. S_ℓ is <u>not invertible</u>

<u>Remark</u> T is <u>left invertible</u> if there exists an operator $L \in \mathcal{B}(\mathcal{H})$ s.t. $LT = I$. It is <u>right invertible</u> if there exist $R \in \mathcal{B}(\mathcal{H})$ s.t. $TR = I$.

↳ S_ℓ is right invertible, S_r is left invertible

<u>THM</u> Let $T \in \mathcal{B}(\mathcal{H})$. Then

(i) T is left invertible $\Leftrightarrow$ $\operatorname{Ker} T = \{0\}$ (injective) and $\operatorname{Ran} T$ is closed

(ii) T is right invertible $\Leftrightarrow$ T is surjective.

<u>Notation</u>: $\varrho(T)$ is the resolvent set of T.

If $\lambda \in \varrho(T)$, then the operator

$$R_\lambda(T) := (\lambda I - T)^{-1} \in \mathcal{B}(\mathcal{H})$$ is the <u>resolvent</u> of T at λ.

$\boxed{\text{THM } \varrho(T) \text{ is nonempty open set in } \mathbb{C}.}$

<u>Proof.</u> If $|\lambda| > \|A\|$, then $\|\frac{A}{\lambda}\| < 1 \Rightarrow \|\cancel{I - \frac{A}{\lambda}}\| < 1$ ~~Corollary~~

$\lambda I - A = \lambda(I - \frac{A}{\lambda})$ is invertible (Neumann series) ~~$I - \frac{A}{\lambda}$ is invertible~~

$$\Downarrow$$

$$\lambda \in \varrho(A) \Longrightarrow \varrho(A) \supseteq \{\lambda \in \mathbb{C} : |\lambda| > \|A\|\}$$

$$\Downarrow$$

$$\varrho(A) \neq \emptyset$$

Suppose that $\lambda_0 \in \varrho(A)$. We show, that if $|\lambda - \lambda_0| \leq \|R_{\lambda_0}(A)\|^{-1}$, then $\lambda I - A$ has a bounded inverse $\Rightarrow \lambda \in \varrho(A)$.

~~$\lambda I - A = \lambda_0 - A$~~

$$\lambda I - A = \lambda_0 I - A - (\lambda_0 - \lambda) I = \underbrace{[I - (\lambda_0 - \lambda)(\lambda_0 I - A)^{-1}]}_{\text{invertible, because } |\lambda_0 - \lambda| < \|(\lambda_0 - A)^{-1}\|^{-1}} (\lambda_0 I - A)$$

$$\hookrightarrow (\lambda I - A)^{-1} = (\lambda_0 I - A)^{-1} [I - (\lambda_0 - \lambda)(\lambda_0 I - A)^{-1}]^{-1} =$$

$$= (\lambda_0 I - A)^{-1} \left(\sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n (\lambda_0 I - A)^{-n} \right)$$

$$= \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n (\lambda_0 I - A)^{-(n+1)} \qquad (*)$$

$\circ$!

Example   Let U be an unitary operator.

$$\|U\|^2 = \|\underbrace{U^*U}_{I}\| = 1 \quad \rightsquigarrow \text{ if } \lambda \in \mathbb{C} \text{ s.t. } |\lambda| > 1$$

by the Neumann series

$$(\lambda I - U)^{-1} = \frac{1}{\lambda}\underbrace{\left(I + \frac{1}{\lambda}U + \frac{1}{\lambda^2}U^2 + \dots\right)}_{\text{bounded operator}} \Rightarrow \lambda \in \rho(U)$$

Similarly for $U^* \rightsquigarrow \lambda \in \rho(U^*)$ if $|\lambda| > 1$.

~~As~~ $\cancel{U - \lambda I = \lambda U}$

As $\quad \lambda I - U = \lambda U\left(U^* - \frac{1}{\lambda} I\right) \Rightarrow \lambda I - U$ is invertible if $0 < |\lambda| < 1 \rightsquigarrow \lambda \in \rho(U)$ $\left(\rightsquigarrow \left|\frac{1}{\lambda}\right| > 1\right)$

Summarizing: if $|\lambda| \neq 1$, then $\lambda \in \rho(U)$.

THM (Resolvent identities)

① Let $\lambda, \mu \in \rho(A)$, then

$$R_\lambda(A) - R_\mu(A) = (\lambda - \mu) R_\lambda(A) R_\mu(A) \qquad \text{1st. Resolvent identity}$$

② For $T, S \in \mathcal{B}(\mathcal{H})$, $\lambda \in \rho(T) \cap \rho(S)$

$$R_\lambda(T) - R_\lambda(S) = R_\lambda(T)(T - S)R_\lambda(S) \qquad \text{2nd Resolvent identity.}$$

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Proof. (1)

$$R_\lambda(A) - R_\mu(A) = (\lambda I - A)^{-1} - (\mu I - A)^{-1} =$$

$$= (\lambda I - A)^{-1} \Big[\underbrace{\mu I - A - (\lambda I - A)}_{(\mu - \lambda) I} \Big] (\mu I - A)^{-1} =$$

$$= (\mu - \lambda) R_\lambda(A) R_\mu(A)$$

(2)

$$R_\lambda(T) - R_\lambda(S) = (\lambda I - T)^{-1} - (\lambda I - S)^{-1} =$$

$$= (\lambda I - T)^{-1} \left[\lambda - S - (\lambda - T) \right] (\lambda I - S)^{-1} =$$

$$= R_\lambda(T) \left[T - S \right] R_\lambda(S)$$

$\circ$!

THM For $T \in \mathcal{B}(\mathcal{H})$, the resolvent set can not be the entire complex plane.

Proof. Let suppose w contrary that $\rho(T) = \mathbb{C}$ i.e. $R_\lambda(T)$ is exist for all $\lambda \in \mathbb{C}$

Claim for any $x, y \in \mathcal{H}$, the map $\lambda \longmapsto \langle x, R_\lambda(T) y \rangle$ is bounded and analytic.

we have seen:

$$(\lambda I - T)^{-1} = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n (\lambda_0 I - A)^{-(n+1)}$$

for $\lambda_0 \in \rho(T)$

$$\Rightarrow \langle x, R_\lambda(T) y \rangle = \sum_{n=0}^{\infty} \langle x, R_{\lambda_0}(T)^{n+1} y \rangle (\lambda_0 - \lambda)^n$$

convergent power series in $(\lambda_0 - \lambda)$ around $\lambda_0 \in \rho(T)$

$\Downarrow$

it is analytic.

$$|\langle x, R_\lambda(T)y\rangle| \leq \|x\|\cdot\|y\|\cdot\|R_\lambda(T)\| \leq \|x\|\cdot\|y\| \frac{1}{|\lambda|}\sum_{n=0}^{\infty}\left\|\frac{T}{\lambda}\right\|^n =$$

$$= \|x\|\cdot\|y\|\cdot \frac{1}{|\lambda|-\|T\|}$$

$\hookrightarrow$ the function is bounded

Recall from the complex functions:

<u>Liouville's Thm</u>: A bounded $\underset{\text{analytic everywhere.}}{\underset{\cap}{\underline{\text{entire}}}}$ function is constant.

$$|\langle x, R_\lambda(T)y\rangle| \leq \|x\|\cdot\|y\|\cdot\frac{1}{|\lambda|-\|T\|} \xrightarrow[|\lambda|\to\infty]{} 0$$

$$\Downarrow \text{ Liouville}$$

$$\langle x, R_\lambda(T)y\rangle = 0 \quad \forall\, x,y\in\mathcal{H},\ \forall\,\lambda\in\mathbb{C}$$

$$\Downarrow$$

$$R_\lambda(T) = 0 \quad \forall\,\lambda\in\mathbb{C} \quad : \lightning \quad !$$

<u>Remarks</u>

① From the first resolvent identity:

$$\frac{R_\lambda(A)-R_\mu(A)}{\lambda-\mu} = -R_\lambda(A)R_\mu(A) \xrightarrow[\lambda\to\mu]{} -R_\mu^2(A)$$

it gives formulas for the derivative of the resolvent.

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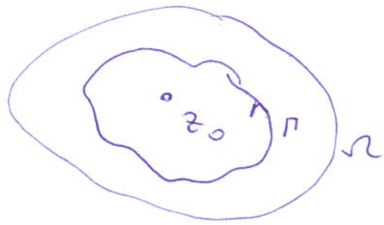
② <u>Recall:</u> Cauchy's integral formula:

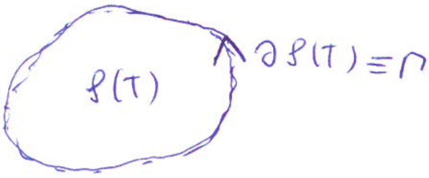
$\Omega \subset \mathbb{C}$ open, $f : \Omega \to \mathbb{C}$ is analytic, $z_0 \in \Omega$

$$f(z_0) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)}{z - z_0}\, dz$$



$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{\Gamma} \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

<u>Similarly:</u>



f is analytic

$$\Downarrow$$

$$f(T) = \frac{1}{2\pi i} \oint_{\partial\rho(T)} f(z)\, R_z(T)\, dz.$$

<u>Riesz – Dunford Theory</u>

<u>Def:</u> $T \in B(\mathcal{H})$, $\sigma(T) := \mathbb{C} \setminus \rho(T)$ is called <u>the spectrum of T</u>.

<u>We have seen:</u> $\rho(T)$ is non-empty, open and cannot be the whole $\mathbb{C}$

and $\rho(T) \subseteq \{\lambda \in \mathbb{C} : |\lambda| > \|T\|\}$

$$\Downarrow$$

$\boxed{\text{THM } T \in B(\mathcal{H}), \text{ then the spectrum } \sigma(T) \text{ of } T \text{ is a nonempty compact subset of } \mathbb{C} \text{ with } \sigma(T) \subset \{\lambda : |\lambda| \le \|T\|\}}$

<u>THM</u> $T \in B(\mathcal{H})$, then the spectrum $\sigma(T)$ of T is a <u>nonempty compact subset</u> of $\mathbb{C}$ with

$$\sigma(T) \subset \{\lambda : |\lambda| \le \|T\|\}$$

Def.

$$\boxed{r(T) := \sup\{|\lambda| : \lambda \in \sigma(T)\}}$$ Spectral Radius of T

THM (Spectral Radius Formula) $T \in \mathcal{B}(\mathcal{H})$

$$\boxed{r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}}$$

Subdivision of the spectrum

$\sigma(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not invertible}\}$

- $\sigma_p(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not injective}\} \equiv \{\lambda \in \mathbb{C} : \operatorname{Ker}(\lambda I - T) \neq \{0\}\}$

point spectrum of T

$\operatorname{Ker}(\lambda I - T) \neq \{0\} \Rightarrow \exists\, x \neq 0,\ x \in \mathcal{H}$ st

$$(\lambda I - T)x = 0 \iff Tx = \lambda x$$

$\Rightarrow$ the elements of $\sigma_p(T)$ are called eigenvalues of T and x ($Tx = \lambda x$) is the eigenvector of T corresponding to λ.

- $\sigma_c(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is injective},\ \underbrace{\operatorname{Ran}(\lambda I - T) \neq \mathcal{H},\ \overline{\operatorname{Ran}(\lambda I - T)} = \mathcal{H}}\}$

continuous spectrum of T

$(\lambda I - T)$ is not surjective, but $\operatorname{Ran}(\lambda I - T)$ is dense in $\mathcal{H}$.

(Recall. T is left invertible $\iff$ T is injective and Ran T is closed
T is right invertible $\iff$ T is surjective)

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- $\sigma_r(T) := \{\lambda \in \mathbb{C} : \lambda I - T$ is injective, $\overline{\mathrm{Ran}(\lambda I - T)} \neq \mathcal{H}\}$

<u>residual spectrum</u> of T.

$\Rightarrow$ $\sigma_p(T)$, $\sigma_c(T)$ and $\sigma_r(T)$ are disjoint subsets of $\sigma(T)$ and

$$\boxed{\sigma(T) = \sigma_p(T) \cup \sigma_c(T) \cup \sigma_r(T)}$$

<u>Remark:</u> There are different subdivision of the spectrum too.

<u>Example:</u> Let consider the left shift operator on l_2

$$S_e : l_2 \to l_2 \quad , \quad S(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$$

- <u>point spectrum:</u> eigenequation:

$$S_e x = \lambda x \quad \text{ie}$$

$$(x_2, x_3, \ldots) = (\lambda x_1, \lambda x_2, \ldots)$$

$\hookrightarrow$
$$\left.\begin{aligned} x_2 &= \lambda x_1 \\ x_3 &= \lambda x_2 \\ x_4 &= \lambda x_3 \\ &\vdots \end{aligned}\right\}$$

choosing $x_1 = 1 \Rightarrow x_2 = \lambda, x_3 = \lambda^2, \ldots$

$x = (1, \lambda, \lambda^2, \ldots)$ is an algebraic solution

we need $x \in l^2$ ie $\sum_{n=0}^{\infty} |\lambda|^{2n} < \infty$

$\Downarrow$

it is convergent if ~~conv~~ ~~radius of the convergence~~

$$\sup \sqrt[n]{|\lambda|^{2n}} = |\lambda|^2 < 1 \qquad \cancel{R = \sup \sqrt[n]{|\lambda|}}$$

$$\Rightarrow \boxed{|\lambda| < 1} \Rightarrow \boxed{\sigma_p(S_e) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}}$$

point spectrum

As $\sigma(S_e)$ is closed and for the spectral radius we have

$$r(S_e) \leq \| S_e \| = 1$$

↳ the spectrum:

$$\sigma(S_e) = \{ \lambda \in \mathbb{C} : |\lambda| \leq 1 \}$$

<u>Question</u>: the points $|\lambda| = 1$ are the parts of the continuous or the residual spectrum?

If $\{ \lambda x - S_e x : x \in l_2 \}$ is not dense in l_2, then there exists $y \in l_2$, $y \neq 0$ s.t. $\langle \lambda x - S_e x, y \rangle = 0$ for all $x \in l_2$

$$\Updownarrow$$

$$\langle x, (\lambda I - S_e)^* y \rangle = 0$$

we know: $S_e^* = S_r$ the right shift

$$\langle x, (\bar{\lambda} I - S_r) y \rangle = 0 \quad \forall x \in l_2$$

$$\Updownarrow$$

$y \in \mathrm{Ker}(\bar{\lambda} I - S_r)$ ie $S_r y = \bar{\lambda} y$, so y is the eigenvector of the right shift

But we have seen, that S_r has not got any eigenvector

$$\Downarrow$$

$\cancel{\{\lambda x - S_e}x$ $\mathrm{Ran}(\lambda I - S_e)$ is dense in $\mathcal{H}$

$$\Downarrow$$

$$\sigma_c(S_e) = \{ \lambda \in \mathbb{C} : |\lambda| = 1 \}$$

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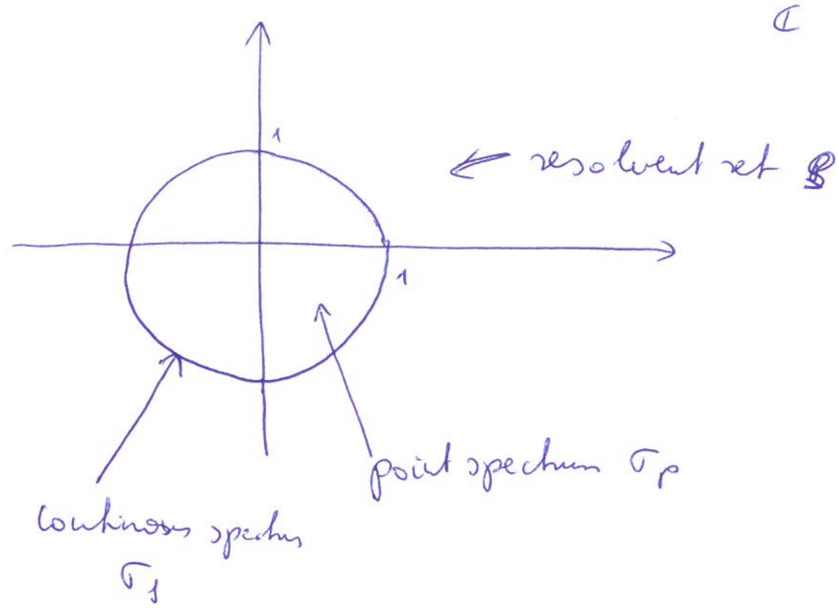
For the left shift operator S_ℓ we have



<u>Recall:</u> $\boxed{\operatorname{Ker} T^* = (\operatorname{Ran} T)^\perp \qquad T \in \mathcal{B}(\mathcal{H})}$

<u>Prop.</u> $T \in \mathcal{B}(\mathcal{H}) \Rightarrow \sigma(T^*) = \overline{\sigma(T)} = \{\lambda \in \mathbb{C} : \bar{\lambda} \in \sigma(T)\}$ and

$$R_\lambda(T)^* = R_{\bar{\lambda}}(T^*) \qquad \lambda \in \varrho(T).$$

<u>Proof.</u> follows from the definition

<u>THM</u>

i) $\lambda \in \sigma_p(T) \Longrightarrow \lambda \in \sigma_p(T^*) \cup \sigma_r(T^*)$

ii) $\lambda \in \sigma_r(T) \Longrightarrow \lambda \in \sigma_p(T^*)$

<u>Proof.</u> (i) $\lambda \in \sigma_p(T) \Longrightarrow \operatorname{Ker}(\lambda I - T) \neq \{0\} \Rightarrow \overline{\operatorname{Ran}(\lambda I - T^*)} =$

$$= \operatorname{Ran}(\lambda I - T^*)^{\perp\perp} = \operatorname{Ker}(\lambda I - T)^\perp$$

$$\neq \mathcal{H}$$

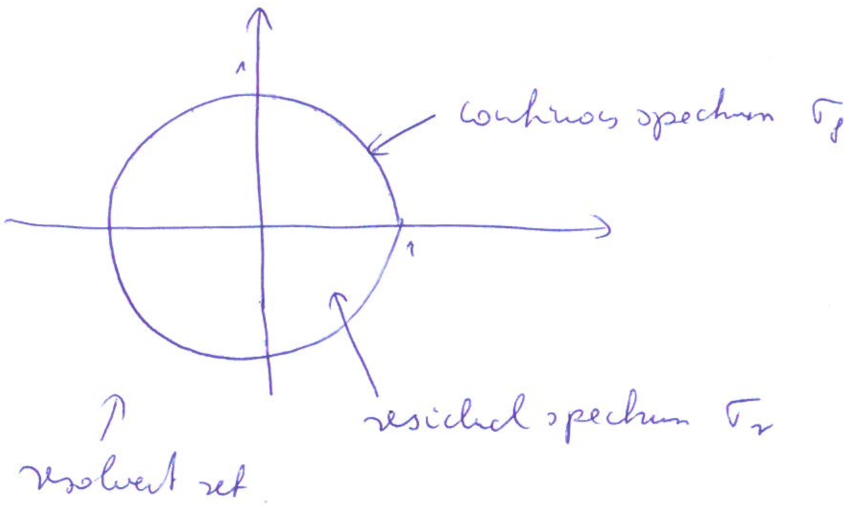
$$\Downarrow$$

$$\lambda \in \sigma_p(T^*) \text{ or } \lambda \in \sigma_r(T^*)$$

ii) If $\lambda \in \sigma_r(T) \Rightarrow \overline{\operatorname{Ran}(\lambda I - T)} \neq \mathcal{H}$

but $\overline{\operatorname{Ran}(\lambda I - T)} = \mathcal{H} \Longleftrightarrow (\lambda I - T)^* = \bar{\lambda} - T^*$ is not injective $\Rightarrow \lambda \in \sigma_p(T^*)$

$\Rightarrow$ For the right shift operator $S_r = S_\ell^*$ we have



<u>Lemma.</u> Let $T \in B(\mathcal{H})$ be a <u>normal operator</u> $(T^*T = TT^*)$.
Then $\lambda \in \rho(T)$ iff there exists $c > 0$ s.t.

$$\|(\lambda - T)x\| \geq c\|x\| \quad \forall x \in \mathcal{H}.$$

$\lambda - T \equiv \lambda I - T$

<u>Proof.</u> If $\lambda \in \rho(T)$, then $c = \|R_\lambda(T)\|^{-1}$ is OK.

~~As a cont~~ For the converse:

$$\|(\lambda - T)x\| \geq c\|x\| \quad \forall x \in \mathcal{H} \Rightarrow \operatorname{Ker}(\lambda - T) = \{0\}$$

As for a normal operator $\|(\lambda - T)x\| = \|(\bar{\lambda} - T^*)x\|$

$$\Downarrow$$

$$\operatorname{Ker}(\bar{\lambda} - T^*) = \{0\}$$

$$\Downarrow \quad \operatorname{Ker}(\bar{\lambda} - T^*) = \operatorname{Ran}(\lambda - T)^\perp$$

$\operatorname{Ran}(\lambda - T)$ is dense

but $\operatorname{Ran}(\lambda - T)$ is closed too $\Rightarrow$ $\lambda \in \rho(T)$. $\square$!

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With the unit circle:

$$\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$$

THM If U is unitary then $\sigma(U) = \mathbb{T}$.

Proof. U is normal and

$$\|(\lambda - U)x\|^2 = \|Ux\|^2 + |\lambda| \cdot \|x\|^2 - 2 \operatorname{Re} \langle Ux, \lambda x \rangle \geq (1 - |\lambda|^2) \|x\|^2$$

$$\Downarrow$$

we can use the lemma for $|\lambda| \neq 1$

$$\Downarrow$$

if $|\lambda| \neq 1$ then $\lambda \in \rho(U)$ $\circ$!

THM Let $A \in \mathcal{B}(\mathcal{H})$ be selfadjoint, and define

$$M := \sup\{\langle Ax, x \rangle : \|x\| = 1\}$$

$$m := \inf\{\langle Ax, x \rangle : \|x\| = 1\}.$$ Then

$$\sigma(A) \subset [m, M] \text{ and } \sigma_r(A) = \emptyset.$$

Proof. For $\lambda, \mu \in \mathbb{R}$

$$\|[(\lambda + i\mu)I - A]x\|^2 = \|(A - \lambda I)x\|^2 + \mu^2 \|x\|^2 - 2\mu \operatorname{Re} i \langle (A - \lambda)x, x \rangle$$

$$= \|(A - \lambda I)x\|^2 + \mu^2 \|x\|^2 \geq \mu^2 \|x\|^2$$

if $\mu \neq 0 \overset{\text{lemma}}{\Longrightarrow} \lambda + i\mu \in \rho(A) \Rightarrow \boxed{\sigma(A) \subset \mathbb{R}}$

If $\lambda > M$, then for any unit vector x

$$\|(\lambda - A)x\| \geq \langle (\lambda - A)x, x \rangle \geq \lambda - M \overset{\text{lemma}}{\Longrightarrow} \lambda \in \rho(A)$$

similarly for $\lambda < m \Rightarrow \sigma(A) \subset [m, M]$

if $\lambda \in \mathbb{R}$ st $\operatorname{Ran}(\lambda I - A)$ is not dense, then

$$\operatorname{Ran}(\lambda I-A)^{\perp} \underset{A=A^*}{=} \operatorname{Ker}(\lambda I-A) \neq \{0\}$$

$$\Downarrow$$

$$\lambda \in \sigma_p(A)$$

$\circ$!

THM Let $T \in \mathcal{B}(\mathcal{H})$ selfadjoint. Then

(i) all eigenvalues of T are real

(ii) eigenvectors corresponding to distinct eigenvalues are orthogonal.

Proof. we have seen $\sigma(T) \subset \mathbb{R} \Rightarrow$ (i) ✓

(ii) Let $Tx_1 = \lambda_1 x_1$, $Tx_2 = \lambda_2 x_2$ $\lambda_1 \neq \lambda_2$

$$\hookrightarrow \langle Tx_1, x_2 \rangle = \lambda_1 \langle x_1, x_2 \rangle = \langle x_1, Tx_2 \rangle \underset{\lambda_2 \in \mathbb{R}}{=} \lambda_2 \langle x_1, x_2 \rangle$$

$$\hookrightarrow \underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} \langle x_1, x_2 \rangle = 0 \Longrightarrow \langle x_1, x_2 \rangle = 0$$

$\circ$!

Def. The set of all eigenvectors corresponding to one particular eigenvalue λ is called the eigenvalue space of λ.

The dimension of that space is called the geometric multiplicity of λ.

An eigenvalue of multiplicity one is called simple or nondegenerate, otherwise it called multiple or degenerate.

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The spectrum of a compact operator

X, Y stands for infinite-dimensional Banach spaces

recall: Riesz lemma

$\mathcal{M} \subset X$ proper closed subspace, then $\forall \varepsilon > 0$

$\exists$ $x \in X$ unit vector s.t. $\operatorname{dist}(x, \mathcal{M}) \geq 1 - \varepsilon$.

If $\mathcal{M}$ is finite dimensional $\exists$ x unit s.t. $\operatorname{dist}(x, \mathcal{M}) = 1$.

THM $A \in K(X, Y) \Rightarrow \operatorname{Ran} A$ is separable

If $\operatorname{Ran} A$ is closed, then it is finite dimensional.

Proof $S_n := \{x \in X : \|x\| < n\} \Rightarrow A(S_n)$ is precompact

Every compact metric space is separable $\Rightarrow A(S_n)$ is separable

$\Downarrow$

their countable union is separable

$$\bigcup_{n=1}^{\infty} A(S_n) = \operatorname{Ran} A$$

Open Mapping Thm: if $\operatorname{Ran} A$ is closed, then A is an open map.

$\Rightarrow$ $A(S_n)$ is open precompact set in $\operatorname{Ran} A$

Every point in $\operatorname{Ran} A$ belongs to some $A(S_n)$

$\Downarrow$

$\operatorname{Ran} A$ is locally compact

$\Downarrow$

finite dimensional

$\circ$!

Cor. $A \in \mathcal{K}(X)$ and $\lambda \in \mathbb{C}$, $\lambda \neq 0$

$\Rightarrow$ $\ker(\lambda I - A)$ is finite dimensional.

Proof.
$N := \ker(\lambda I - A)$ is closed for any linear operator

if $\lambda \neq 0$, $\boxed{A(N) \subseteq N}$

$\cancel{x \in N \rightsquigarrow Ax = \lambda x \rightsquigarrow (\lambda I - A)x = 0}$

So if A is compact, then N is finite dimensional by the THM.

Remark $\dim X = \infty$. If $A \in K(X) \Rightarrow$ A cannot invertible

$\Downarrow$

$\boxed{0 \in \sigma(A)}$

Prop. $A \in K(X) \Rightarrow$ The point spectrum $\sigma_p(A)$ is countable and has only one possible limit point 0.

Prop. $A \in K(X)$. If $\lambda \neq 0$ and $\lambda \in \sigma(A) \Rightarrow \lambda \in \sigma_p(A)$

<u>THM</u> (Riesz–Schauder)

Let A be a compact operator. Then

(i) $\sigma(A)$ is a countable set containing 0

(ii) No point other than 0 can be the limit point of $\sigma(A)$

(iii) Each nonzero point of $\sigma(A)$ is an eigenvalue of A and has finite multiplicity

(iV) For any $\lambda \in \mathbb{C}$ $\operatorname{Ran}(\lambda I - A)$ is closed.

<u>THM</u> (Hilbert)

Let T be a compact selfadjoint operator on $\mathcal{H}$. Then

(i) For any non-zero eigenvalue λ, the eigenspace of λ is finite dimensional subspace

(ii) If the distinct eigenvalues sequence $(\lambda_n)_{n\in\mathbb{N}}$ converges to λ^*, then $\lambda^* = 0$.

(iii) $\exists$ $v_1, v_2 \ldots$ ONB in $\mathcal{H}$ consisting of eigenvectors of T and for all $x \in \mathcal{H}$

$$\boxed{Tx = \sum_{n=1}^{\infty} \lambda_n \langle x, v_n \rangle v_n}$$

where $Tv_n = \lambda_n v_n$

!