GAUSSIAN MARKOV TRIPLETS

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The Markov property of states on algebras of the canonical commutation relation is studied and in the case of Gaussian states several equivalent properties are obtained. The detailed description is given in terms of a block matrix. The relation to classical multivariate Gaussian Markov triplets is also described. The minimizer of relative entropy with respect to a Gaussian Markov state has the Markov property.

Keywords: Weyl unitaries, von Neumann entropy, CCR algebra, Markov triplet, Gaussian state, relative entropy.

1. Introduction

A simple, but important example of a stochastic process with dependence is one in which each random variable depends on the one preceding it and is conditionally independent of all the other preceding random variables. Such a process is said to be Markov. A stochastic process is Markov if and only if the Shannon entropy increase is constant. Multivariate normal distributions are described by a positive definite matrix and if their joint distribution is Gaussian as well then it can be represented by a block matrix. The aim of this note is to study Markov triplets (Markov process with three random variables) by using the block matrix technique. A Markov triplet is characterized by the form of its block covariance matrix and by the form of the inverse of this matrix. We also study some non-commutative analogues, namely the CCR algebraic case. The notion of Gaussian state was developed in the framework of the C*-algebraic approach to the canonical commutation relation (CCR) [17,12,5,19]. The CCR-algebra is generated by the Weyl unitaries (satisfying a commutation relation, therefore Weyl algebra is an alternative terminology). The Gaussian states on CCR algebras can be regarded as analogues of Gaussian distributions in classical probability: The n-point functions can be computed from the 2-point functions and in a kind of central limit theorem the limiting state is Gaussian and it maximizes the von Neumann entropy when the 2-point function is fixed [20]. The Gaussian states are quite tractable, for example the von Neumann entropy has an explicit expression [7,5].

The Markov property was invented by Accardi in the non-commutative (or quantum probabilistic) setting [1,2]. (Another approach is in the paper [9].) This Markov property is based on a completely positive, identity preserving map, so-called quasi-conditional expectation and it was formulated in the tensor product of matrix algebras. A state of a tensor product system is Markovian if and only if the von Neumann entropy increase is constant. This property and a possible definition of the Markov condition was suggested in [21]. A remarkable property of the von Neumann entropy is the strong subadditivity [16,11,18,23] which plays an important role in the investigations of quantum system's correlations. The above mentioned constant increase of the von Neumann entropy is the same as the equality for the strong subadditivity of von Neumann entropy.

A CCR (or Weyl) algebra is parametrized by a Hilbert space, we use the notation $CCR(\mathcal{H})$ when $\mathcal{H}$ is the Hilbert space. Assume that φ_{123} is a state on the composite system $CCR(\mathcal{H}_1)\otimes CCR(\mathcal{H}_2)\otimes CCR(\mathcal{H}_3)$. Denote by $\varphi_{12}, \varphi_{23}$ the restriction to the first two and to the second and third factors, similarly φ_2 is the restriction to the second factor. The Markov property is defined as

$$S(\varphi_{123}) - S(\varphi_{12}) = S(\varphi_{23}) - S(\varphi_2),$$

where S denotes the von Neumann entropy [18]. When φ_{123} is Gaussian, it is given by a positive operator (corresponding to the 2-point function) and the main goal of the present paper is to describe the Markov property in terms of this operator. The paper [25] studies a similar question for the CAR algebra. Although the multivariate Gaussian case (in classical probability) is rather different from the present non-commutative setting, we used the same block matrix formalism. A Gaussian state is described by a block matrix and the Markov property is formulated by the entries. A Markovian Gaussian state induces multivariate Gaussian restrictions, but they are very special in that framework.

2. Classical Markov triplets

Let $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3$ be a finite set with probability distribution $p(x_1, x_2, x_3)$ $(x_i \in \mathcal{X}_i, 1 \le i \le 3)$. The Markov property is defined by conditional probabilities in the stochastic setting:

$$p(x_3|x_1,x_2)=p(x_3|x_2)$$

which means

$$\frac{p(x_1,x_2,x_3)}{p(x_1,x_2)}=\frac{p(x_2,x_3)}{p(x_2)}, \tag{1}$$

and we say that the random variables X_1,X_2 and X_3 form a Markov chain, denoted by $X_1 \to X_2 \to X_3$. We remark that $X_1 \to X_2 \to X_3$ implies $X_3 \to X_2 \to X_1$. The computation

$$p(x_1,x_3|x_2)=\frac{p(x_1,x_2,x_3)}{p(x_2)}=\frac{p(x_1,x_2)p(x_3|x_2)}{p(x_2)}=p(x_1|x_2)p(x_3|x_2)$$

shows that the Markov property holds if and only if X_1 and X_3 are conditionally independent. If X_3 has the interpretation as "future", X_2 is the "present" and X_1 is the "past", then having the Markov property means that, given the present state, future state is independent of the past state. In other words, the description of the present state fully captures all the information that could influence the future evolution of the process. In order to move to the C*-algebra setting, we denote by $\mathcal{A}_{123}$ the algebra of functions on $\mathcal{X}$. The subalgebra $\mathcal{A}_{12}$ consists of those functions of the variables x_1,x_2,x_3 whose values are actually do not depend on x_3. The subalgebras $\mathcal{A}_{23}$ and $\mathcal{A}_2$ are defined similarly and $\mathcal{A}_2=\mathcal{A}_{12}\cap\mathcal{A}_{23}$.

The conditional expectation $E_{12}^{123}:\mathcal{A}_{123}\to\mathcal{A}_{12}$ defined as

$$E_{12}^{123}(g)(x_1,x_2)=\int g(x_1,x_2,x_3)\frac{p(x_1,x_2,x_3)}{p(x_1,x_2)}\,dx_3. \tag{2}$$

This leaves fixed any functions $h\in\mathcal{A}_{12}$ and $E_{12}^{123}(hg)=hE_{12}^{123}(g)$ for every $g\in\mathcal{A}_{123}$. Similarly,

$$E_2^{23}(f)(x_2)=\int f(x_2,x_3)\frac{p(x_2,x_3)}{p(x_2)}\,dx_3. \tag{3}$$

If (1) holds, then

$$E_{12}^{123} \text{ restricted to } \mathcal{A}_{23} \text{ is } E_2^{23}. \tag{4}$$

It is natural to investigate the Markov chains from information theoretical point of view. The Shannon entropy $H(X_1,X_2,\dots,X_k)$ of a k-tuple of discrete random variables with values in $\mathcal{X}$ is

$$H(X_1,X_2,\dots,X_k)=-\sum_x p(x)\log p(x),$$

where summation is over $x \in \mathcal{X}^k$. The Shannon entropy has many properties that agree with the intuitive notion of what a measure of information should be, for example it helps us to express the dependence among the random variables. One can check easily that

$$H(X_1, X_2, X_3) - H(X_2, X_3) = H(X_1, X_2) - H(X_2)$$

if and only if $X_1 \to X_2 \to X_3$. Further details can be found in [6,23].

3. Gaussian Markov triplets

To show an important classical example we investigate the multivariate normal distribution forming a Markov chain. All these results with proofs can be found in [3]. Let $\mathbf{X} := (\mathbf{X_1}, \mathbf{X_2}, \ldots, \mathbf{X_n})$ be an n-tuple of real or complex random variables. The (i, j) element of the n x n covariance matrix is given by

$$C_{i,j} := \mathbb{E}(X_i \overline{X_j}) - \mathbb{E}(X_i)\mathbb{E}(\overline{X_j}),$$

where $\mathbb{E}$ denotes the expectation. The covariance matrix is positive semidefinite. The mean $\mathbf{m} := (\mathbf{m_1}, \mathbf{m_2}, \ldots, \mathbf{m_n})$ consists of the expectations $m_i = \mathbb{E}(X_i)$, $(i = 1, 2, \ldots, n)$. Let M be a positive definite $n \times n$ matrix and $\mathbf{m}$ be a vector. Then

$$f_{\mathbf{m},\mathbf{M}}(\mathbf{x}) := \sqrt{\frac{\det M}{(2\pi)^n}} \exp\left(-\frac{1}{2}\left\langle(\mathbf{x}-\mathbf{m}), M(\mathbf{x}-\mathbf{m})\right\rangle\right) \tag{5}$$

is a multivariate Gaussian probability distribution, denoted by $N(m, M^{-1})$, with expectation $\mathbf{m}$ and with quadratic matrix M. If $\mathbf{m} = \mathbf{0}$, then we write simply $f_M(\mathbf{x})$. If M is diagonal, then (5) is the product of functions of one-variable which means the independence of the random variables. It is a remarkable fact that the covariance matrix of the distribution (5) is M^{-1}. The following lemma is well known, see for example [3,10].

Lemma 3.1. *Let*

$$M = \begin{bmatrix} A & B \\ B^* & D \end{bmatrix} \tag{6}$$

be a positive definite $(m + k)$ by $(m + k)$ matrix written in block matrix form. Then the marginal of the Gaussian probability distribution

$$f_M(\mathbf{x}_1, \mathbf{x}_2) := \sqrt{\frac{\det M}{(2\pi)^{(m+k)}}} \exp\left(-\frac{1}{2}\langle(\mathbf{x}_1, \mathbf{x}_2), M(\mathbf{x}_1, \mathbf{x}_2)\right)\rangle$$

on $\mathbb{R}^m$ *is the distribution*

$$f_1(\mathbf{x}_1) := \sqrt{\frac{\det M}{(2\pi)^m \det D}} \exp\left(-\frac{1}{2}\langle \mathbf{x}_1, (A - BD^{-1}B^*)\mathbf{x}_1\rangle\right). \tag{7}$$

The matrix $(M|D) := A - BD^{-1}B^*$ appears often in the matrix analysis, and called the Schur complement of D in M [26].

Now turn to the conditional distributions. Given the random variables $\mathbf{X}_1$ and $\mathbf{X}_2$, the conditional density is given by

$$f(\mathbf{x}_2|\mathbf{x}_1) := \frac{f_{(\mathbf{x}_1,\mathbf{x}_2)}(\mathbf{x}_1,\mathbf{x}_2)}{f_{\mathbf{x}_1}(\mathbf{x}_1)}, \tag{8}$$

which is a function of $\mathbf{x}_2$, since $\mathbf{x}_1$ is fixed. If $(\mathbf{X}_1, \mathbf{X}_2)$ is Gaussian with a quadratic matrix (6), then the conditional distribution (8) is the Gaussian $N(-D^{-1}B^*\mathbf{x}_1, D^{-1})$.

Let $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$ be random variables with joint probability distribution $f(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$. The distribution of the appropriate marginals are $f(\mathbf{x}_1, \mathbf{x}_2)$, $f(\mathbf{x}_2, \mathbf{x}_3)$ and $f(\mathbf{x}_2)$. In accordance with the foregoing $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$ is called a Markov triplet if

$$f(\mathbf{x}_3|\mathbf{x}_1, \mathbf{x}_2) = f(\mathbf{x}_3|\mathbf{x}_2). \tag{9}$$

We use the notation $\mathbf{X}_1 \to \mathbf{X}_2 \to \mathbf{X}_3$ for the Markov triplets as before. Let $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$ be a Gaussian random variable with the quadratic matrix

$$M = \begin{bmatrix} A_1 & A_2 & B_1 \\ A_2^* & A_3 & B_2 \\ B_1^* & B_2^* & D \end{bmatrix} \tag{10}$$

and with expectation $\mathbf{m} = 0$. The next theorem gives the characterization of Gaussian Markov triplets [3].

Theorem 3.1. *For the Gaussian triplet* $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$ *with quadratic matrix (10) and with expectation 0, the following conditions are equivalent*

(i) $\mathbf{X}_1 \to \mathbf{X}_2 \to \mathbf{X}_3$

(ii) $B_1 = 0$

(iii) The conditional distribution $f(\mathbf{x}_3|\mathbf{x}_1, \mathbf{x}_2)$ *does not depend on* $\mathbf{x}_1$.

(iv) The covariance matrix of $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$ *is of the form*

$$\begin{bmatrix} S_{11} & S_{12} & S_{12}S_{22}^{-1}S_{23} \\ S_{12}^* & S_{22} & S_{23} \\ S_{23}^*S_{22}^{-1}S_{12}^* & S_{23}^* & S_{33} \end{bmatrix}. \tag{11}$$

(v) $h(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3) - h(\mathbf{X}_2, \mathbf{X}_3) = h(\mathbf{X}_1, \mathbf{X}_2) - h(\mathbf{X}_2)$, *where* $h(\mathbf{X}) := -\int f(\mathbf{x}) \log f(\mathbf{x}) \mathrm{d}\mathbf{x}$ *is the Boltzmann-Gibbs entropy.*

4. Markov states

Recall that a pair $(\mathcal{A}, \varphi)$ consisting of an C*-algebra $\mathcal{A}$ and its state φ is called an algebraic probability space. An algebraic random variable is an embedding $j : \mathcal{B} \to \mathcal{A}$ of an algebra $\mathcal{B}$ into $\mathcal{A}$. The state $\varphi \circ j$ of $\mathcal{B}$ is called the distribution of the random variable j. At the characterization of the Markov property in a non-commutative C*-algebra setting, the basic problem is that in the quantum setting conditional probabilities and conditional expectations (preserving a given state) do not exist. Indeed, states compatible with norm one projections tend to be trivial in the extremely noncommutative case, i.e. factorial case, that is algebras with trivial center.

Assume that $\mathcal{A}$ is the tensor product of matrix algebras: $\mathcal{A}_{123} = \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \mathcal{A}_3$. A state φ of $\mathcal{A}_{123}$ is called Markov if there exists a completely positive unital mapping $F : \mathcal{A}_{123} \to \mathcal{A}_1 \otimes \mathcal{A}_2 \equiv \mathcal{A}_{12}$ such that

$$F(A_1 \otimes I_{23}) = A_1 \otimes I_2$$

and

$$\varphi(A_{123}) = \varphi(F(A_{123})).$$

The first condition tells that $\mathcal{A}_1$ is in the fixed point algebra of F and the second one says that F preserves the given state. (Originally, Accardi and Frigerio called F quasi-conditional expectation and wrote about infinite tensor product [2].) If φ is a faithful, normal state with a density operator D, $(D > 0,\ \mathrm{Tr} D = 1)$, then its von Neumann entropy is given by

$$S(\varphi) \equiv S(D) = -\mathrm{Tr} D \log D.$$

Theorem 4.1. *The following conditions are equivalent, $\varphi \equiv \varphi_{123}$.*

(i) φ *is Markov.*

(ii) The von Neumann entropy increase is constant, i.e.

$$S(\varphi_{123}) - S(\varphi_{23}) = S(\varphi_{12}) - S(\varphi_2). \tag{12}$$

(iii) There exist a subalgebra $\mathcal{B}$ and a conditional expectation from $\mathcal{A}_{123}$ onto $\mathcal{B}$ such that

$$\mathcal{A}_1 \subset \mathcal{B} \subset \mathcal{A}_{12}$$

and E leaves the state φ invariant.

(iv) There is a state transformation

$$\mathcal{E} : \mathcal{A}_2 \to \mathcal{A}_{23}$$

such that $(Id_1 \otimes \mathcal{E})(\varphi_{12}) = \varphi_{123}$.

For the details, see Chapter 9 of [23]. Moreover, it was shown that a state is Markov if and only if it is the convex combination of orthogonal product type states [11].

Theorem 4.1 inspire us to call a triplet $(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$ of subalgebras of a C*-algebra $\mathcal{A}$ to Markovian with respect to the state φ of $\mathcal{A}$ if $\mathcal{A}_3$ and $\mathcal{A}_1$ are conditionally independent with respect to $\mathcal{A}_2$ and (12) holds.

5. CCR algebra and Gaussian states

Let $\mathcal{H}$ be a Hilbert space. Assume that for every $f \in \mathcal{H}$ a unitary operator $W(f)$ is given such that the relations

$$W(f_1)W(f_2) = W(f_1 + f_2)\exp(\mathrm{i}\,\sigma(f_1, f_2)),$$

$$W(-f) = W(f)^*$$

hold for $f_1, f_2, f \in \mathcal{H}$ with $\sigma(f_1, f_2) := \mathrm{Im}\langle f_1, f_2\rangle$. The C*-algebra generated by these unitaries is unique and denoted by $CCR(\mathcal{H})$ [19,27].

The C*-algebra $CCR(\mathcal{H})$ is not separable, but nuclear [8], therefore its tensor product with any other C*-algebra is uniquely defined [15]. The relations show that $W(f_1)$ and $W(f_2)$ commute if f_1 and f_2 are orthogonal. It follows that $CCR(\mathcal{H}_1) \otimes CCR(\mathcal{H}_2)$ is isomorphic to $CCR(\mathcal{H}_1 \oplus \mathcal{H}_2)$.

The C*-algebra $CCR(\mathcal{H})$ has a very natural state

$$\omega(W(f)) := \exp\left(-\|f\|^2/2\right) \tag{13}$$

which is called Fock state. The GNS-representation of $CCR(\mathcal{H})$ is called Fock representation and it leads to the the Fock space $\mathcal{F}(\mathcal{H})$ with cyclic vector Φ. Since ω is actually a product state, the GNS Hilbert space is a tensor product. We shall identify the abstract unitary $W(f)$ with the representing unitary acting on the Fock space $\mathcal{F}(\mathcal{H})$. The map

$$t \mapsto \Phi(W(tf))$$

is a strongly continuous 1-parameter group of unitaries and according to the Stone theorem we have

$$W(tf) = \exp(\mathrm{i}tB(f)) \quad \text{and} \quad \frac{\partial}{\partial t}\bigg|_{t=0} W(tf) = \mathrm{i}B(f)$$

for a self-adjoint operator $B(f)$, called field operator. The distribution of a field operator is Gaussian with respect to the Fock state.

The Fock state (13) can be generalized by choosing a positive operator $A \in B(\mathcal{H})$:

$$\omega_A(W(f)) := \exp\left(-\|f\|^2/2 - \langle f, Af\rangle\right). \tag{14}$$

This is called Gaussian or quasi-free state. By derivation we get

$$\begin{aligned}\omega_A(B(f)B(g)) &= -\mathrm{i}\sigma(f,g) + \frac{1}{2}\left(\langle f,(I+2A)g\rangle + \langle g,(I+2A)f\rangle\ \right)\\ &= \mathrm{Re}\,\langle f,(I+2A)g\rangle - \mathrm{iIm}\,\langle f,g\rangle\ ,\end{aligned}$$

and all higher order correlation functions are expressed by this two-point functions [5]. Moreover,

$$\omega_A(B^+(f)B^-(g)) = \langle g, Af\rangle\,. \tag{15}$$

For $0 \le A \in B(\mathcal{H})$, the the statistical operator of the Gaussian state ω_A of $CCR(\mathcal{H})$ in the Fock representation is

$$\rho_A := \frac{\Gamma(A(I+A)^{-1}))}{\mathrm{Tr}\Gamma(A(I+A)^{-1}))},$$

where Γ is the second quantization of operators. If λ_i are the eigenvalues of A, then ρ_A has the eigenvalues

$$\prod_i \frac{1}{1+\lambda_i}\left(\frac{\lambda_i}{1+\lambda_i}\right)^{n_i},$$

where $n_i \in \mathbb{Z}_+$. Therefore the von Neumann entropy is

$$S(\omega_A) = \mathrm{Tr}\kappa(A),$$

where $\kappa(t) = -t\log t + (t+1)\log(t+1)$.

Assume that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ and write the positive mapping $A \in B(\mathcal{H})$ in the form of block matrix:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{bmatrix}.$$

If $f \in \mathcal{H}_1$, then

$$\omega_A(W(f\oplus 0)) = \exp\left(-\|f\|^2/2 - < f, A_{11}f\right)\ .$$

Therefore the restriction of the Gaussian state ω_A to $CCR(\mathcal{H}_1)$ is the Gaussian state $\omega_{A_{11}}$.

6. Markov property on CCR

Let $\mathcal{H}_{123} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ be a finite dimensional Hilbert space and consider the CCR-algebras $\mathcal{A}_i := CCR(\mathcal{H}_i)$. Then $\mathcal{A}_{123} := CCR(\mathcal{H}_{123}) = \mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \mathcal{A}_3$ holds. Assume that D_{123} is a statistical operator in $\mathcal{A}_{123}$ and we denote by D_{12}, D_2, D_{23} its reductions into the subalgebras $\mathcal{A}_{12}, \mathcal{A}_2, \mathcal{A}_{23}$, respectively. As before, these subalgebras form a Markov triplet with respect to the state D_{123} if

$$S(D_{123}) - S(D_{23}) = S(D_{12}) - S(D_2), \tag{16}$$

where S denotes the von Neumann entropy and we assume that both sides are finite in the equation. Actually, the strong additivity of von Neumann entropy (16) is strongly related to statistical sufficiency and we have further equivalent conditions.

(i) $D_{123}^{\mathrm{it}} D_{23}^{-\mathrm{it}} = D_{12}^{\mathrm{it}} D_2^{-\mathrm{it}}$ for every real t [13,22].
(ii) $D_{123}^{1/2} D_{23}^{-1/2} = D_{12}^{1/2} D_2^{-1/2}$.
(iii) $\log D_{123} - \log D_{23} = \log D_{12} - \log D_2$.
(iv) There are positive matrices $X, Y \in \mathcal{A}_{12}$ and $0 \le Z \in \mathcal{A}_{123}$, such that $D_{123} = XZ$, $D_{12} = YZ$ and the commutation relation $ZX = XZ$ and $ZY = YZ$ hold.

Remark that some of the equivalences are valid also in infinite dimensional Hilbert space, for example, the equivalence of (i) and (iv) is obtained in [13]. Mostly we study the Markov property of a Gaussian state $\omega_A \equiv \omega_{123}$ with a density matrix D_{123}, where A is a positive operator acting on $\mathcal{H}_{123}$, given in a block matrix form

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}. \tag{17}$$

Then the restrictions D_{23}, D_{12} and D_2 are also Gaussian states with the positive operators

$$D = \begin{bmatrix} I & 0 & 0 \\ 0 & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{bmatrix}, \quad B = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & I \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} I & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & I \end{bmatrix},$$

respectively. Then the equivalent conditions above implies the following [14].

Theorem 6.1. *For a Gaussian state ω_A the Markov property (16) is equivalent to the condition*

$$A^{it}(I+A)^{-it} D^{-it}(I+D)^{it} = B^{it}(I+B)^{-it} C^{-it}(I+C)^{it} \tag{18}$$

for every real t.

Moreover, we can give also a description in terms of the operator A.

Theorem 6.2. *A Gaussian state ω_A is Markov if and only if A has the following form*

$$A = \begin{bmatrix} A_{11} & [a\,0] & 0 \\ \begin{bmatrix} a^* \\ 0 \end{bmatrix} & \begin{bmatrix} c\,0 \\ 0\,d \end{bmatrix} & \begin{bmatrix} 0 \\ b \end{bmatrix} \\ 0 & [0\,b^*] & A_{33} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} A_{11}\,a \\ a^*\,c \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} d & b \\ b^* & A_{33} \end{bmatrix} \end{bmatrix},$$

where the parameters a, b, c, d (and 0) are matrices.

In other words A should be a block diagonal matrix. There are nontrivial Markovian Gaussian states which are not a product in the time localization. However, the first and the third subalgebras are always independent.

Theorem 6.3. *Let $\omega \equiv \omega_A$ be a Markovian Gaussian state on the CCR-algebra $\mathcal{A}_{123}$. If ψ is the state minimizing the relative entropy $S(\psi||\omega_A)$ under the constraint that $\psi|\mathcal{A}_1 = \varphi$ is fixed, then ψ is a Markov state.*

The proof and a similar result are in the paper [24]. In the probabilistic case the similar statatement is well-known, see [4], for example.

7. Connection to classical Gaussian

Let $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ be vector-valued random variables with Gaussian joint probability distribution

$$\sqrt{\frac{\det M}{(2\pi)^n}} \exp\left(-\frac{1}{2}\langle \mathbf{x}, M\mathbf{x}\rangle\right),$$

where M is positive definite matrix. Theorem 3.1 says that the triplet $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$ has the Markov property if and only if the covariance matrix $S = M^{-1}$ of $(\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$ is of the form (11), that is

$$S_{13} = S_{12} S_{22}^{-1} S_{23}, \tag{19}$$

see [3]. To show some analogy between the classical Gaussian and the CCR Gaussian case, we can formulate a somewhat similar description to (19) in the CCR setting [24].

Theorem 7.1. *The block matrix*

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

gives a Gaussian state with the Markov property if and only if

$$A_{13} = A_{12} f(A_{22}) A_{23}$$

for any continuous function $f : \mathbb{R} \to \mathbb{R}$.

This shows that the CCR condition is much more restrictive than the classical one. Next we assume that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ and assume that $\dim \mathcal{H}_i = k$ $(1 \le i \le 3)$. Choose an orthonormal basis $\{f_j : 1 \le j \le 3k\}$ such that

$$f_{ik+r} \in \mathcal{H}_i, \qquad 1 \le i \le 3, \quad 0 \le r \le k-1$$

and unit vectors e_j, $1 \le j \le 3k$ such that

$$e_{ik+r} \in \mathcal{H}_i, \qquad 1 \le i \le 3, \quad 0 \le r \le k-1$$

and

$$\langle e_t, e_u \rangle \text{is real for any } 1 \le t, u \le 3k.$$

This implies that the Weyl unitaries $W(te_j) = \exp(\mathrm{i}tB(e_j))$ commute and the field operators $B(e_j)$ have a joint distribution. If ω_A is Gaussian, then the classical (multi-valued) Gaussian triplet

$$(B(e_1), \dots, B(e_k)), \quad (B(e_{k+1}), \dots, B(e_{2k})), \quad (B(e_{2k+1}), \dots, B(e_{3k})) \tag{20}$$

has Gaussian joint distribution with covariance $S^*(I + 2A)S$, where S is defined as $Sf_j = e_j$, $1 \le j \le 3k$. Moreover, Theorem 7.1 and 3.1 implies the following result [14].

Theorem 7.2. *If* ω_A *is a Gaussian Markov state, then the classical (multi-valued) Gaussian triplet (20) is Markovian as well, moreover,*

$$(B(e_1), \dots, B(e_k)) \quad \textit{and} \quad (B(e_{2k+1}, \dots, B(e_{3k}))$$

are independent.

The converse is not true as some numerical computation shows [14]. However, if for every λA $(\lambda > 0)$ the classical Markov property is true, then the Markovianity of the CCR Gaussian ω_A follows [24].

8. Conclusions

A simple, but important example of a stochastic process with dependence is one in which each random variable depends on the one preceding it and is conditionally independent of all the other preceding random variables. In this paper we attended to the case of three random variables, called Markov triplet. In the classical probability three random variables form a Markov triplet if and only if we have got equality in the strong subadditivity of Shannon entropy for their distributions. This fact yields to define Markov property for a state on a CCR algebra as a state which saturate the strong subadditivity of von Neumann entropy with equality. We investigated two important special cases: the multivariate Gaussian distribution in classical probability and its analogue on CCR algebras, the so-called quasi-free states. Both of them is completely characterized by a positive matrix (quadratic or covariance matrix in the classical case and the so-called symbol in the CCR case) and we characterized the Markov property via these matrices. We found that the CCR case is much more restrictive: a CCR quasi-free Markov state implies classical Gaussian Markov triplets if we consider commuting Weyl unitaries, while the converse is not true in general.

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