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Markovian quasifree states on canonical anticommutation relation algebras

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The characterization of quasifree product states on CAR algebras is given. We also prove that the quasifree states on CAR algebra which saturate the strong additivity of von Neumann entropy with equality are product states. © *2007 American Institute of Physics.* [DOI: 10.1063/1.2817817]

I. INTRODUCTION

The class of quasifree states on canonical anticommutation relation (CAR) algebras can be regarded as analogs of Gaussian distributions in classical probability in that all the n-point functions can be computed from the two-point functions. This fact allows us to study systematically the complicated correlations that can occur on spin chains; particularly, the quasifree states are quite tractable. The von Neumann entropy has an explicit expression by its symbol.[4,19]

A remarkable property of the von Neumann entropy is the strong subadditivity, which plays an important role in the investigations of quantum system's correlations.[4,12,14] It has been shown that the strong subadditivity of the von Neumann entropy is tightly related to the Markov property invented by Accardi[1] and Accardi and Frigerio.[2] This noncommutative Markov property is based on a completely positive, identity-preserving map, so-called quasiconditional expectation. A state is called a Markov state if it is compatible with such a map. A state of a threefold tensor-product system is Markovian if and only if it takes the equality for the strong subadditivity of von Neumann entropy which is referred to as a strong additivity of the von Neumann entropy.[10] A similar equivalence relation of the Markov property and the strong additivity of the von Neumann entropy was shown for even states recently[13] for CAR algebras and for more general graded quantum systems.

As the quasifree states on CAR algebras are even states, it is natural to pose the question about the necessary and sufficient conditions for the strong additivity of the von Neumann entropy because these condition will be necessary and sufficient for the Markovianity, as we mentioned above. Our goal is to answer this question in this paper.

The present paper is organized as follows. After a preliminary section containing some crucial results on CAR algebra, on quasifree states, and on the connection between the Markov states and the strong additivity of the von Neumann entropy, we give the complete characterization of the quasifree product states. Section IV contains our main result: We show that every quasifree Markov state is a product state.

II. PRELIMINARIES

A. The CAR algebra

In this section, we summarize known properties of the algebra of the canonical anticommutation relation. The works References 5 and 8 contain all that we need. Consider the Hilbert space

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$l^2(\mathbb{Z})$ with the canonical orthonormal basis $\{\delta_k : k\in\mathbb{Z}\}$. For $I\subset\mathbb{Z}$, let $\mathcal{A}(I)$ be the CAR algebra corresponding to the linear subspace spanned by $\{\delta_k : k\in I\}$. This is a unital C^*-algebra generated by the elements $\{a_i : i\in I\}$ satisfying the anticommutation relations

$$a_i a_j + a_j a_i = 0,$$

$$a_i a_j^* + a_j^* a_i = \delta_{i,j}\mathbf{1},$$

for $i,j\in I$. The operators a^* and a are often called *creator* and *annihilator*, respectively. It is easy to see that $\mathcal{A}(I)$ is the linear span of the identity and monomials of the form

$$A_{i(1)}A_{i(2)}\cdots A_{i(k)}, \tag{1}$$

where $i(1)<i(2)<\ldots<i(k)$ and each factor $A_{i(j)}$ is one of the four operators $a_{i(j)}$, $a_{i(j)}^*$, $a_{i(j)}a_{i(j)}^*$, and $a_{i(j)}^*a_{i(j)}$. The CAR algebra $\mathcal{A}$ is defined by

$$\mathcal{A}\equiv \overline{\vee_{l\in L}\mathcal{A}(\{l\})}^{C^*}.$$

It is known that for $I=\{1,2,\ldots,n\}$, $\mathcal{A}(I)$ is isomorphic to a matrix algebra $M_{2^n}(\mathbb{C})\simeq M_2(\mathbb{C})^1 \otimes\cdots\otimes M_2(\mathbb{C})^n$. An explicit isomorphism is given by the so-called Jordan-Wigner isomorphism. Namely, the relations

$$e_{11}^{(i)} := a_i a_i^*,\quad e_{12}^{(i)} := V_{i-1}a_i,$$

$$e_{21}^{(i)} := V_{i-1}a_i^*,\quad e_{22}^{(i)} := a_i^* a_i,$$

$$V_i := \prod_{j=1}^{i}(I-2a_j^*a_j)$$

determine a family of mutually commuting 2×2 matrix units for $i\in I$. Since

$$a_i = \prod_{j=1}^{i-1}(e_{11}^{(j)}-e_{22}^{(j)})e_{12}^{(i)},$$

the above matrix units generate $\mathcal{A}(I)$ and give an isomorphism between $\mathcal{A}(I)$ and $M_2(\mathbb{C})\otimes\cdots\otimes M_2(\mathbb{C})$:

$$e_{i_1j_2}^{(1)}e_{i_2j_2}^{(2)}\ldots e_{i_nj_n}^{(n)} \leftrightarrow e_{i_1j_1}\otimes e_{i_2j_2}\otimes\cdots\otimes e_{i_nj_n}. \tag{2}$$

[Here, e_{ij} stands for the standard matrix units in $M_2(\mathbb{C})$.] It follows from this isomorphism that $\mathcal{A}(I)$ has a unique tracial state τ.

Let $J\subset\mathbb{Z}$. There exists a unique automorphism Θ_J of $\mathcal{A}(\mathbb{Z})$ such that

$$\Theta_J(a_i) = -a_i \quad\text{and } \Theta_J(a_i^*) = -a_i^* \quad (i\in J),$$

$$\Theta_J(a_i) = a_i \quad\text{and } \Theta_J(a_i^*) = a_i^* \quad (i\notin J).$$

In particular, we write Θ instead of $\Theta_{\mathbb{Z}}$. Θ_J is inner; i.e., there exists a v_J self-adjoint unitary in $\mathcal{A}(J)$ given by

$$v_J\equiv\prod_{i\in J}v_i,\quad v_i\equiv a_i^*a_i - a_i a_i^*, \tag{3}$$

such that $\Theta_J(A)=(\mathrm{Ad}\,v_J)A\equiv v_J A v_J^*$ for any $a\in\mathcal{A}(J)$. The *odd* and *even parts* of $\mathcal{A}_I$ are defined as

$$\mathcal{A}(I)^+ := \{A \in \mathcal{A}(I):\Theta_I(A) = A\}, \quad \mathcal{A}(I)^- := \{A \in \mathcal{A}(I):\Theta_I(A) = -A\}. \tag{4}$$

$\mathcal{A}(I)^+$ is a subalgebra but $\mathcal{A}(I)^-$ is not. The graded commutation relation for CAR algebras is known: If $A \in \mathcal{A}(K)$ and $B \in \mathcal{A}(L)$, where $K \cap L = \emptyset$, then

$$AB = \epsilon(A,B)BA, \tag{5}$$

where

$$\epsilon(A,B) = \begin{cases} -1 & \text{if } A \text{ and } B \text{ are odd} \\ +1 & \text{otherwise.} \end{cases}$$

The parity automorphism is the special case of the action of the *gauge group* $\{\alpha_\theta : 0 \leq \theta < 2\pi\}$ with

$$\alpha_\theta(a_i) = e^{-i\theta} a_i.$$

An element $a \in \mathcal{A}$ is *gauge invariant* if $\alpha_\theta(a) = a$ for all $0 \leq \theta < 2\pi$.

B. Quasifree states

In classical probability, a Gaussian measure leads to a characteristic function which is the exponential of a quadratic form. Its logarithm is therefore a quadratic polynomial, and all correlations beyond the second order vanish. In the CAR case for even states, it is also possible to define the useful concept of the correlation function (cumulants or truncated function in other words) (see Ref. 8 for details). Baslev and Verbeure[7] defined a quasifree state of the CAR algebra to be an even state for which the correlation functions vanish for $n \geq 3$. For gauge-invariant states, this is equivalent to the earlier definition of Shale and Stinespring,[18] which is the following. The gauge-invariant quasifree state ω_Q on $\mathcal{A}$ is given by

$$\omega_Q(a^*_{i_1} \cdots a^*_{i_m} a_{j_n} \cdots a_{j_1}) := \delta_{mn} \det([Q_{i_k j_l}]^n_{k,l=1}), \tag{6}$$

where Q is an operator on $\mathcal{H} = l^2(\mathbb{Z})$ with $0 \leq Q \leq \mathbf{1}$ and $Q_{ij} = \langle \delta_i, Q\delta_j \rangle$ are the matrix elements of Q in the standard basis of $\mathcal{H}$. We can see that for a gauge-invariant quasifree state, the n-point function is wholly determined by these two-point functions. The operator Q is called the *symbol of the state* ω_Q.

For example, suppose an interaction-free fermionic system described by a Hamiltonian H with discrete spectrum $\{h_i\}$, such that $e^{-\beta \hat{H}}$ is a trace-class operator for $\beta > 0$, where $\hat{H}$ is the second-quantized Hamiltonian. The Gibbs state ρ_β of the system in the Fock representation has a density operator $e^{-\beta \hat{H}} / \mathrm{Tr}\, e^{-\beta \hat{H}}$, and its value on monomials can be expressed as

$$\rho_\beta[a^*(x_1) \cdots a^*(x_m) a(y_n) \cdots a(y_1)] := \delta_{mn} \det [\langle x_i, Q y_j \rangle]^n_{i,j=1}, \tag{7}$$

where Q has the same eigenvectors as H with corresponding eigenvalues $e^{-\beta h_i} / \mathrm{Tr}\, e^{-\beta h_i}$. So, the quasifree states can be considered as the generalization of canonical Gibbs states for noninteracting systems.

We collect some facts about the quasifree states. All these results with proofs can be found in Refs. 4, 8, 9, and 19. Obviously, the quasifree states are even. A quasifree state is pure if and only if its symbol is a projection. Let P_n be the projection from $l^2(\mathbb{Z})$ onto the finite-dimensional subspace spanned by $\{\delta_1, \ldots, \delta_n\}$. The restriction of ω_Q to the subalgebra $\mathcal{A}[P_n l^2(\mathbb{Z})]$ is again quasifree with symbol $P_n Q P_n$. The quasifree state ω_Q is translation invariant if and only if its symbol Q is a Töplitz operator; i.e., there exists a sequence $\{q_k : k \in \mathbb{Z}\}$ such that $Q_{lk} = q(k-l)$. Quasifree states are quite tractable: The von Neumann entropy of a quasifree state ω_Q has an explicit expression by its symbol

$$S(\omega_Q) = \mathrm{Tr}[-Q \log Q - (\mathbf{1} - Q)\log(\mathbf{1} - Q)]. \tag{8}$$

C. The strong subadditivity of the von Neumann entropy and the Markov property

Let ρ be a density matrix on a Hilbert space $\mathcal{H}$, i.e., $\mathrm{Tr}\,\rho=1$, $\rho\geqslant 0$. A remarkable property of the von Neumann entropy

$$S(\rho) = -\,\mathrm{Tr}\,\rho \log \rho \tag{9}$$

is the strong subadditivity (SSA)

$$S(\rho_{12}) + S(\rho_{23}) \geqslant S(\rho_{123}) + S(\rho_2), \tag{10}$$

which was proven first by Lieb and Ruskai[12] for a tripartite state ρ_{123} on the system $\mathcal{H}\equiv\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3$, where ρ_{12}, ρ_{23}, and ρ_2 are the reduced density matrices on $\mathcal{H}_{12}\equiv\mathcal{H}_1\otimes\mathcal{H}_2$, $\mathcal{H}_{23} \equiv \mathcal{H}_2\otimes\mathcal{H}_3$, and $\mathcal{H}_2$, respectively. Now, let us turn to the CAR case. However, when the set I is countable, the CAR algebra is isomorphic to the C^*-infinite tensor product $\overline{\otimes_I M_2(\mathbb{C})}^{C^*}$ as we have seen, but the isomorphism does not preserve the natural localization. The elements of the disjoint subsystems do not commute in contrast to the tensor-product case. In spite of these difficulties, the strong subadditivity of the von Neumann entropy also holds for CAR algebras, as was proven by Araki and Moriya.[5] It is interesting to find the states which saturate the SSA with equality. The following theorem of Petz[15,16] gives an equivalent condition for the equality (see Ref. 11 for a survey).

Theorem 1: *Assume that ρ_{123} is invertible. The equality holds in the strong subadditivity of entropy if and only if the following equivalent conditions hold:*

(1) $\log \rho_{123} - \log \rho_{23} = \log \rho_{12} - \log \rho_2$,
(2) ${\rho_{123}}^{it}{\rho_{23}}^{-it} = {\rho_{12}}^{it}{\rho_2}^{-it} (t \in \mathbb{R})$.

□

In the theorem above, $\mathcal{B}(\mathcal{H}_{12})$, $\mathcal{B}(\mathcal{H}_{23})$, and $\mathcal{B}(\mathcal{H}_2)$ are considered as subalgebras of $\mathcal{B}(\mathcal{H})$ via the relations $A_{12}\otimes I_3$, $I_1\otimes A_{23}$, and $I_1\otimes A_2\otimes I_3$, respectively, where $A_{12}\in\mathcal{B}(\mathcal{H}_{12})$, $A_{23}\in\mathcal{B}(\mathcal{H}_{23})$, and $A_2\in\mathcal{B}(\mathcal{H}_2)$, and I_k denotes the identity on $\mathcal{H}_k$, $(k=1,2,3)$. In von Neumann's unifying scheme for classical and quantum probability, an important ingredient was missing: *conditioning*. In order to study nontrivial statistical dependences, particularly to construct Markov processes, this gap had to be filled. Accardi and Frigerio[2] proposed the following definition.

Definition 2: Consider a triplet $\mathcal{C}\subset\mathcal{B}\subset\mathcal{A}$ of unital C^*-algebras. A quasiconditional expectation with respect of the given triplet, is a completely positive, identity-preserving linear map $E:\mathcal{A}\to\mathcal{B}$ such that

$$E(ca) = cE(a), \quad a \in \mathcal{A}, c \in \mathcal{C}.$$

We also have

$$E(ac) = E(a)c, \quad a \in \mathcal{A}, c \in \mathcal{C}$$

as E is a real map. Now we are able to give the definition of the *quantum Markov state* due to Accardi and co-workers.[1–3]

Definition 3: A state ϕ on $\mathcal{A}$ is called a (short) quantum Markov state if there exists a quasiconditional expectation E with respect to the triplet $\mathcal{C}\subset\mathcal{B}\subset\mathcal{A}$, satisfying

(i) $\phi_{\mathcal{B}}\circ E = \phi$,
(ii) $E(\mathcal{A}\backslash\mathcal{C}) \subset (\mathcal{B}\backslash\mathcal{C})$,

where $\phi_{\mathcal{B}}$ denotes the restriction of ϕ onto the subalgebra $\mathcal{B}$.

Here, the subalgebras $\mathcal{C}$, $\mathcal{B}$, and $\mathcal{A}$ symbolize the past, the present, and the future, respectively. For the (i) part of the definition, we also say that the state ϕ is compatible with map E. Condition (ii) is the Markov property which possesses the usual interpretation: The future does not depend on the past but only the present.

It has been shown that the Markovianity is tightly related to the strong subadditivity of the von Neumann entropy. Namely, a state of a three-composed tensor-product system is a short Markov state if and only if it takes the equality for the strong subadditivity inequality of entropy.[10,15] Moreover, a translation invariant quantum Markov state of the quantum spin algebra has a constant entropy increment at each step by the strong additivity (see Proposition 11.5 in Ref. 14). A similar result for CAR algebras was shown for even states recently;[13] namely, an even state on a CAR algebra is the Markov state[3] if and only if it saturates the strong subadditivity of the von Neumann entropy with equality.

As the quasifree states are even states, if we find the necessary and sufficient conditions for the strong additivity of the von Neumann entropy, we also have the complete characterization of the Markov quasifree states on the CAR algebra. This is the program of this paper.

III. QUASIFREE PRODUCT STATES

For a pair of disjoint subsets I_1 and I_2 of $\mathbb{Z}$, let ϕ_1 and ϕ_2 be given states of the CAR algebras $\mathcal{A}(I_1)$ and $\mathcal{A}(I_2)$, respectively. If for a state ϕ of the joint system $\mathcal{A}(I_1 \cup I_2)$ [which coincides with the C^*-algebra generated by $\mathcal{A}(I_1)$ and $\mathcal{A}(I_2)$]

$$\phi(A_1A_2) = \phi_1(A_1)\phi_2(A_2) \tag{11}$$

holds for all $A_1 \in \mathcal{A}(I_1)$ and all $A_2 \in \mathcal{A}(I_2)$, then ϕ is called a *product state* extension of ϕ_1 and ϕ_2 [or with respect to the subalgebras $\mathcal{A}(I_1)$ and $\mathcal{A}(I_2)$]. We will use the notation $\phi=\phi_1 \circ \phi_2$. For an arbitrary (finite or infinite) number of given subsystems, a product state extension is shown to exist if and only if all states of subsystems except at most one are even.[6] In this section, we investigate the product states in the quasifree case.

Since any restriction of a quasifree state is again quasifree, it is clear that if a quasifree state is a product state it must be a product of quasifree states. Let $Q \in M_n(\mathbb{C})$ be a positive contraction, the symbol of the gauge-invariant CAR quasifree state ω_Q. Denote $I=(i_1, \ldots, i_k)$ and $J=(j_1, \ldots, j_l)$ two disjoint ordered subsets of the index set $K=(1, \ldots, n) \subset \mathbb{Z}$, such that $I \cup J = K$, i.e., $k+l=n$. Let P_I be the projection from $\mathbb{C}^n$ onto the finite-dimensional subspace spanned by the subset of the canonical orthonormal basis $\{\delta_{i_1}, \ldots, \delta_{i_k}\}$. Obviously, $P_J=\mathbf{1}_n - P_I$ is a projection onto the orthogonal subspace spanned by $\{\delta_{j_1}, \ldots, \delta_{j_l}\}$. Consider the CAR algebras $\mathcal{A}[P_I l^2(\mathbb{Z})]$ and $\mathcal{A}[P_J l^2(\mathbb{Z})]$ as the subsystems of a bipartite system. The symbols of the restrictions of the quasifree state ω_Q to the subalgebras are given by $Q_I = P_I Q P_I$ and $Q_J = P_J Q P_J$. For the quasifree product states, we show the following.

Theorem 4: *With the conditions above*, ω_Q *is a product state with respect to the subsystems* $\mathcal{A}(I)$ *and* $\mathcal{A}(J)$; i.e., $\omega_Q = \omega_{Q_I} \circ \omega_{Q_J}$ *if and only if for its symbol* Q *the condition*

$$Q_{i_r, j_s} = 0 \tag{12}$$

holds for all $r=1, \ldots, k$ *and* $s=1, \ldots, l$.

Proof: For necessity, let us suppose that ω_Q is a product state; i.e., $\omega_Q(A_I A_J) = \omega_{Q_I}(A_I)\omega_{Q_J}(A_J)$ holds for all $A_I \in \mathcal{A}(I)$ and $A_J \in \mathcal{A}(J)$. As the quasifree states are even states, it must disappear on the products $A_I A_J$ when either A_I or A_J is odd. Particularly,

$$\omega_Q(a_{i_r}^* a_{j_s}) = Q_{i_r, j_s} = 0, \tag{13}$$

as we stated. We show that it is also a sufficient condition. Indeed, this condition makes it possible to arrange Q to a block-diagonal form by interchangements of columns and rows. Suppose that by interchanging the columns N times, we can obtain a matrix in which the indices of the first k columns are $i_1, \ldots, i_k$. Since Q is symmetric, with N appropriate interchangements of the rows we arrive at the following block-diagonal matrix:

$$\begin{bmatrix} Q_I & 0 \\ 0 & Q_J \end{bmatrix}. \tag{14}$$

As we get this matrix by $2N$ interchangements of rows and columns, the determinant did not change. We get

$$\det Q = \det Q_I \det Q_J, \tag{15}$$

which is equivalent with our statement. ■

IV. CHARACTERIZATION OF THE QUASIFREE MARKOV STATES ON CAR ALGEBRAS

Let $Q \in M_n(\mathbb{C})$ be a positive contraction, the symbol of the gauge-invariant quasifree state ω_Q on the CAR algebra $\mathcal{A}[Pl^2(\mathbb{Z})]$, where P is the projection from $l^2(\mathbb{Z})$ onto the finite-dimensional subspace spanned by the subset of the canonical orthonormal basis $\{\delta_1, \ldots, \delta_n\}$. Similarly, let R_1, R_2, and R_3 be the projections from $l^2(\mathbb{Z})$ onto the final-dimensional subspaces spanned by $\{\delta_1, \ldots, \delta_k\}$, $\{\delta_{k+1}, \ldots, \delta_{k+l}\}$, and $\{\delta_{k+l+1}, \ldots, \delta_{k+l+m}\}$, respectively, where $k+l+m=n$. Obviously R_1, R_2, and R_3 are mutually orthogonal projections. The restrictions of ω_Q to the subalgebras $\mathcal{A}_{12} \equiv \mathcal{A}[(R_1 \vee R_2) l^2(\mathbb{Z})]$, $\mathcal{A}_{23} \equiv \mathcal{A}[(R_2 \vee R_3) l^2(\mathbb{Z})]$, and $\mathcal{A}_2 \equiv \mathcal{A}[R_2 l^2(\mathbb{Z})]$ are again quasifree with symbols $Q_{12}=(R_1 \vee R_2)Q(R_1 \vee R_2)$, $Q_{23}=(R_2 \vee R_3)Q(R_2 \vee R_3)$, and $Q_2=R_2QR_2$, respectively. It is useful to write Q in the following block-matrix form.

$$Q = \begin{bmatrix} A & X & Z \\ X^* & B & Y \\ Z^* & Y^* & C \end{bmatrix}, \tag{16}$$

where $A=A^* \in M_k(\mathbb{C})$, $B=B^* \in M_l(\mathbb{C})$, $C=C^* \in M_m(\mathbb{C})$, $X \in M_{k,l}(\mathbb{C})$, $Y \in M_{l,m}(\mathbb{C})$, and $Z \in M_{k,m}(\mathbb{C})$. Here, $M_{k,l}(\mathbb{C})$ denotes the k by l complex matrices, and $M_k(\mathbb{C}) \equiv M_{k,k}(\mathbb{C})$ for simplicity. We define the following auxiliary density matrices on the matrix algebra $M_{2n}(\mathbb{C})$

$$\rho_Q := \frac{1}{n}\begin{bmatrix} Q & 0 \\ 0 & \mathbf{1}_n - Q \end{bmatrix}, \tag{17}$$

$$\rho_{Q_{12}} := \frac{1}{n}\begin{bmatrix} W_{12} & 0 \\ 0 & \mathbf{1}_n - W_{12} \end{bmatrix}, \tag{18}$$

$$\rho_{Q_{23}} := \frac{1}{n}\begin{bmatrix} W_{23} & 0 \\ 0 & \mathbf{1}_n - W_{23} \end{bmatrix}, \tag{19}$$

and, finally,

$$\rho_{Q_2} := \frac{1}{n}\begin{bmatrix} W_2 & 0 \\ 0 & \mathbf{1}_n - W_2 \end{bmatrix}, \tag{20}$$

where $\mathbf{1}_n$ is the identity matrix in $M_n(\mathbb{C})$, and we have used the following notations for the block matrices:

$$W_{12} = \begin{bmatrix} A & X & 0 \\ X^* & B & 0 \\ 0 & 0 & \frac{1}{2}\mathbf{1}_m \end{bmatrix} = \begin{bmatrix} Q_{12} & 0 \\ 0 & \frac{1}{2}\mathbf{1}_m \end{bmatrix}, \tag{21}$$

$$W_{23} = \begin{bmatrix} \frac{1}{2}\mathbf{1}_k & 0 & 0 \\ 0 & B & Y \\ 0 & Y^* & C \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\mathbf{1}_k & 0 \\ 0 & Q_{23} \end{bmatrix}, \tag{22}$$

and, finally,

$$W_2 = \begin{bmatrix} \frac{1}{2}\mathbf{1}_k & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & \frac{1}{2}\mathbf{1}_m \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\mathbf{1}_k & 0 & 0 \\ 0 & Q_2 & 0 \\ 0 & 0 & \frac{1}{2}\mathbf{1}_m \end{bmatrix}. \tag{23}$$

One can check that $\rho_{Q_{12}}$, $\rho_{Q_{23}}$, and ρ_{Q_2} are the reduced density matrices of the state ρ_Q of the composite system $\mathcal{B} \equiv M_{2n}(\mathbb{C})$ to the subalgebras $\mathcal{B}_{12} \equiv M_{k+l}(\mathbb{C}) \oplus \mathbf{1}_m \oplus M_{k+l}(\mathbb{C}) \oplus \mathbf{1}_m$, $\mathcal{B}_{23} \equiv \mathbf{1}_k \oplus M_{l+m}(\mathbb{C}) \oplus \mathbf{1}_k \oplus M_{l+m}(\mathbb{C})$, and $\mathcal{B}_2 \equiv \mathbf{1}_k \oplus M_l(\mathbb{C}) \oplus \mathbf{1}_m \oplus \mathbf{1}_k \oplus M_l(\mathbb{C}) \oplus \mathbf{1}_m$, respectively.

A simple computation shows that the von Neumann entropy of these density matrices can be expressed by the von Neumann entropy of the corresponding quasifree states, namely,

$$S(\rho_Q) = \frac{S(\omega_Q)}{n} + \log n, \tag{24}$$

$$S(\rho_{Q_{12}}) = \frac{S(\omega_{Q_{12}})}{n} + \log n + \frac{m}{n} \log 2, \tag{25}$$

$$S(\rho_{Q_{23}}) = \frac{S(\omega_{Q_{23}})}{n} + \log n + \frac{k}{n} \log 2, \tag{26}$$

$$S(\rho_{Q_2}) = \frac{S(\omega_{Q_2})}{n} + \log n + \frac{k+m}{n} \log 2. \tag{27}$$

With the help of these formulas, the strong subadditivity of the von Neumann entropy of the quasifree states is given by

$$S(\omega_{Q_{12}}) + S(\omega_{Q_{23}}) - S(\omega_Q) - S(\omega_{Q_2}) = n\{S(\rho_{Q_{12}}) + S(\rho_{Q_{23}}) - S(\rho_Q) - S(\rho_{Q_2})\} \geqslant 0.$$

So, we have got equality for the quasifree states if and only if we saturate the equality for the auxiliary states defined above. Remember that Theorem 1 means an equivalent equation for the density matrices. Our goal is to find the necessary and sufficient conditions for the matrix Q for which this equivalent equation holds. By the analytic continuation, the (2) part of Theorem 1 also holds for $t=-i$; i.e.,

$$\rho_Q \rho_{Q_{23}}^{-1} = \rho_{Q_{12}} \rho_{Q_2}^{-1} \tag{28}$$

is a necessary condition for the strong additivity of the von Neumann entropy. If we like to compute the inverse of a block matrix, the following theorem is very useful. Its checking is a simple multiplication, but its constructive proof based on the Schur complement can be found in several books on linear algebra,[17]

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1}+A^{-1}B(D-CA^{-1}B)^{-1}CA^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1} \\ -(D-CA^{-1}B)^{-1}CA^{-1} & (D-CA^{-1}B)^{-1} \end{bmatrix} \tag{29}$$

if A and D are square matrices and A is invertible.

With the help of (29), we have for the inverse

$$\rho_{Q_{23}}^{-1} = n\begin{bmatrix} W_{23}^{-1} & 0 \\ 0 & (\mathbf{1}_n - W_{23})^{-1} \end{bmatrix}, \tag{30}$$

where

$$W_{23}^{-1} = \begin{bmatrix} 2\mathbf{1}_k & 0 & 0 \\ 0 & B^{-1}+B^{-1}YR^{-1}Y^*B^{-1} & -B^{-1}YR^{-1} \\ 0 & -R^{-1}Y^*B^{-1} & R^{-1} \end{bmatrix}.$$

We have used the abbreviation $R=C-Y^*B^{-1}Y=R^*$ for simplicity. Similarly,

$$(\mathbf{1}_n - W_{23})^{-1} = \begin{bmatrix} 2\mathbf{1}_k & 0 & 0 \\ 0 & (\mathbf{1}_l-B)^{-1}+(\mathbf{1}_l-B)^{-1}YS^{-1}Y^*(\mathbf{1}_l-B)^{-1} & -(\mathbf{1}_l-B)^{-1}YS^{-1} \\ 0 & -S^{-1}Y^*(\mathbf{1}_l-B)^{-1} & S^{-1} \end{bmatrix},$$

with $S=(\mathbf{1}_m-C)-Y^*(\mathbf{1}_m-B)^{-1}Y=S^*$. The inverse of ρ_{Q_2} is much more simple, it is a block-diagonal matrix in the form

$$\rho_{Q_2}^{-1} = n\,\mathrm{diag}(2\mathbf{1}_k, B^{-1}, 2\mathbf{1}_m, 2\mathbf{1}_k, (\mathbf{1}_l-B)^{-1}, 2\mathbf{1}_m). \tag{31}$$

Substituting our matrices into Eq. (28), we get that the equality holds if and only if $Y=0$ and $Z=0$, so these conditions are necessary to get equality in the strong subadditivity of the von Neumann entropy.

If we consider the (2) part of Theorem 1 for $t=i$, we get another necessary condition for the equality, namely,

$$\rho_{Q_{23}}\rho_{Q_2}^{-1} = \rho_Q \rho_{Q_{12}}^{-1}. \tag{32}$$

With the help of (29), we can compute the inverses again as we have done above, and after substituting, we get that Eq. (32) holds if and only if $X=0$ and $Z=0$. So, Q must have the following block-diagonal form:

$$Q = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{bmatrix}; \tag{33}$$

i.e., the quasifree state is a product state.

As any product state is a Markov state, we can summarize our results in the following way.

Theorem 5: *A quasifree state on a CAR algebra saturates the strong subadditivity of the von Neumann entropy with equality (or Markov state, equivalently) if and only if it is the product of its reduced states.* □

Since a quasifree state is translation invariant if and only if its symbol is a Töplitz matrix, as a consequence of Theorem 5, we get that a translation-invariant quasifree state is Markov state if and only if its symbol is some constant times the identity matrix.

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