A NOTE ON COMPARATIVE CONVEXITY OF OPERATOR FUNCTIONS WITH RESPECT TO KUBO-ANDO MEANS

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ABSTRACT. We obtain connections between comparative convexity properties and comparative concavity properties of certain functions defined on positive definite operators by continuous functional calculus. Among others, we show that any positive, operator concave function defined on the positive half-line is convex with respect to the harmonic mean, and any positive, operator convex, and numerically non-increasing function on the positive half-line is concave with respect to the harmonic mean. We propose some open problems, as well.

1. INTRODUCTION

A real function f defined on some interval $I \subset \mathbb{R}$ is said to be convex if

$$f\left((1-\lambda)x+\lambda y\right) \leq (1-\lambda)f(x)+\lambda f(y) \qquad \left(x, y \in I, \lambda \in [0,1]\right)$$

holds. For continuous functions, convexity coincides with the a priori weaker mid-point convexity property

$$f\left(\frac{1}{2}(x+y)\right) \leq \frac{1}{2}\left(f(x)+f(y)\right) \qquad \left(x, y \in I\right). \tag{1}$$

In fact, under very mild conditions, convexity is equivalent to mid-point convexity, see [4, Thm. 30]. If we introduce the notation $m_A(x,y) := \frac{1}{2}(x+y)$ for the arithmetic mean, equation (1) can be written as follows:

$$f\left(m_A(x,y)\right) \leq m_A\left(f(x), f(y)\right) \qquad \left(x, y \in I\right). \tag{2}$$

It is quite natural to consider equation (2) with means different from the arithmetic mean. By considering (2) with general means, we arrive to the topic of comparative convexity. For a detailed introduction to this field, we refer to [9, Chapter 2]. However, we collect some of the basic notions of the topic below based on the book [9].

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In order to define comparative convexity of functions, we have to define *means* first.

Definition 1 (Means of real numbers)**.** *Let I be a real interval. A function $m : I \times I \to \mathbb{R}$ is called a* mean *if*

$$\min\{x, y\} \leq m(x, y) \leq \max\{x, y\} \qquad \left(x, y \in I\right)$$

holds.

Comparative convexity is defined for general means as follows.

Definition 2 (Comparative convexity)**.** *Let m and n be means of positive numbers. We say that a function $f : (0,\infty) \to (0,\infty)$ is (m,n)-convex if*

$$f\left(m(x, y)\right) \leq n\left(f(x), f(y)\right) \tag{3}$$

holds for any $x, y \in (0,\infty)$. The (m,n)-concave property is defined accordingly.

Obviously, if both m and n is the arithmetic mean, we recover the classical notion of convexity. If m is the arithmetic mean and n is the geometric mean, we get the notion of *log-convexity.* The terminology is motivated by the fact that the inequality $f\left(m_A(x, y)\right) \leq m_G\left(f(x), f(y)\right)$ is equivalent to the classical convexity of the function $\log \circ f$. (Here and throughout, m_A and m_G stand for the arithmetic and geometric means of positive numbers, respectively.) If both m and n is the geometric mean, we get the notion of *multiplicative convexity.* We emphasize that now we follow the terminology of [9]. Sometimes, (m_A, m_G)-convexity is referred to as multiplicative convexity, see, e.g., [4, Def. 31]. However, in the present paper (m_A, m_G)-convexity is called log-convexity, and (m_G, m_G)-convexity is referred to as multiplicative convexity.

Comparative convexity can be defined analogously for maps sending positive operators to positive operators. The tools we need for the definition are continuous functional calculus, Kubo-Ando operator means, and the order on self-adjoint operators induced by positivity. So, throughout this paper, if f is a continuous function defined on some interval $I \subset \mathbb{R}$ and A is a self-adjoint bounded operator on a Hilbert space such that its spectrum $\sigma(A)$ is contained in I, then the symbol $f(A)$ denotes the operator obtained by continuous functional calculus. In the next section we provide a detailed introduction to the topic of operator means and operator connections. The order on self-adjoint operators induced by positivity (the *Löwner order*) is defined as follows: $A \leq B$ if and only if $B - A$ is positive semidefinite. A continuous function $f : \mathbb{R} \supset I \to \mathbb{R}$ is said to be *operator monotone* if for any self-adjoint bounded operators A and B which satisfy $A \leq B$ and $\sigma(A) \cup \sigma(B) \subset I$ we have $f(A) \leq f(B)$. Operator convexity of continuous functions is defined similarly. If $f : \mathbb{R} \supset I \to \mathbb{R}$ is a continuous function satisfying $f\left(\frac{1}{2}(A+B)\right) \le \frac{1}{2}\left(f(A)+f(B)\right)$ for any self-adjoint operators A and B with $\sigma(A) \cup \sigma(B) \subset I$, then f is said to be *operator convex*. Operator *concavity* is defined accordingly. Note that the above definition are precisely the definitions of mid-point convexity and concavity, but we consider only continuous functions, and in this case, convexity and mid-point convexity coincide.

Let us introduce some notation, as well. Throughout this paper, $\mathscr{H}$ is a complex Hilbert space and $\mathscr{B}(\mathscr{H})$ denotes the set of all bounded linear operators on $\mathscr{H}$. The symbol $\mathscr{B}(\mathscr{H})^{++}$ stands for the set of all positive definite (that is, positive semidefinite and invertible) elements of $\mathscr{B}(\mathscr{H})$.

The main result of this paper concerns comparative convexity of operator functions. In particular, we show that any positive, operator concave function defined on the positive half-line is convex with respect to the harmonic mean, and any positive, operator convex, and numerically non-increasing function on the positive half-line is concave with respect to the harmonic mean.

2. Kubo-Ando means of positive operators

Based on the seminal paper [6], we recall some basic facts from the Kubo-Ando theory of operator means. For the sake of simplicity and clarity, we present the basics of the theory of operator means only for positive definite operators. In this way, we avoid some technical difficulties which are not relevant from our viewpoint.

Definition 3 (Operator connection, operator mean)**.** *A binary operation* $\sigma : \mathscr{B}(\mathscr{H})^{++} \times \mathscr{B}(\mathscr{H})^{++} \to \mathscr{B}(\mathscr{H})^{++}; (A,B) \mapsto A\sigma B$ *is said to be an* operator connection *if*

- *it is monotone in both variables, that is,*
$$(A \le B \,\&\, C \le D) \Longrightarrow A\sigma C \le B\sigma D,$$
- *the equation*
$$C(A\sigma B)C = (CAC)\,\sigma\,(CBC)$$
holds for any $A, B, C \in \mathscr{B}(\mathscr{H})^{++}$,
- *and if the monotone decreasing sequences* A_n *and* B_n *tend to* A *and* B*, respectively, in the strong operator topology, then* $A_n\sigma B_n$ *tends to* $A\sigma B$ *monotone decreasingly in the strong operator topology.*

An operator connection satisfying the equation $I\sigma I = I$ *is called* operator mean. *(The symbol* I *stands for the identity operator on* $\mathscr{H}$*.)*

Example 1. We enumerate some of the most important operator means.

- The *arithmetic* mean of positive definite operators is denoted by ∇ and is defined as $A\nabla B = \frac{1}{2}(A+B)$.
- The geometric mean [1, 10] is denoted by the symbol # and is defined as
$$A\#B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{1/2}A^{1/2}.$$
- The harmonic mean (denoted by the symbol !) is defined as
$$A!B = \left(\frac{1}{2}\left(A^{-1}+B^{-1}\right)\right)^{-1}.$$

There is a very important one-to-one correspondence between operator connections and positive operator monotone functions defined on the positive half-line. This correspondence is as follows. Given an operator connection σ, one can define a positive function g on $(0,\infty)$ by $g(x) := 1\sigma x$. The map $\sigma \mapsto g$ obtained this way is an affine order-isomorphism from the class of connections onto the class of positive operator monotone functions [6, Thm. 3.2.]. Moreover, the connection σ can be recovered from the function g by the formula
$$A\sigma B = A^{\frac{1}{2}}g\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}},$$
see, e.g. [6, Eq. (3.8)]. It follows from the definition of operator means, that the means are in a one-to-one correspondence with the positive operator monotone functions satisfying $g(1)=1$. Sometimes, the function g obtained from the connection σ is referred to as the *representing function* of σ.

Example 2. The representing functions of the arithmetic, the geometric, and the harmonic means are $g_A(x) = (1+x)/2$, $g_G(x) = \sqrt{x}$, and $g_H(x) = (2x)/(1+x)$, respectively.

We introduce the following notation. For an operator monotone function $g : (0,\infty) \to (0,\infty)$, we denote the corresponding operator connection by M_g, that is,
$$M_g(A,B) = A^{\frac{1}{2}}g\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}} \qquad \left(A,B \in \mathscr{B}(\mathscr{H})^{++}\right).$$

2.1. **Comparative operator convexity.** Comparative convexity for functions sending positive operators to positive operators can be defined similarly to the case of functions sending positive numbers to positive numbers. Therefore, the following definition is formally very similar to Definition 2.

Definition 4 (Comparative operator convexity)**.** *Let M and N be operator connections (in the sense of Definition 3). We say that a continuous* *function* $f:(0,\infty)\to(0,\infty)$ *is* (M,N)*-convex if*

$$f(M(X,Y)) \le N\big(f(X), f(Y)\big) \tag{4}$$

holds for any $X,Y \in \mathscr{B}(\mathscr{H})^{++}$. *(Recall that expressions like* $f(X)$ *are defined by continuous functional calculus.) The* (M,N)*-concave property is defined accordingly.*

If a function f is (M,M)-convex (-concave) for some operator connection M then we may say that f is convex (concave) with respect to the connection M.

Remark 1. Note that while in Definition 2 comparative convexity was defined with means, in Definition 4 comparative operator convexity was defined with operator connections (and not only with operator means). In fact, there is no reason to restrict ourselves to operator means in the definition of comparative operator convexity.

2.2. **The adjoint of a connection.** Taking the adjoint of an operator connection is a common operation in the theory of operator connections. In order to define the adjoint of a connection, first we define the adjoint of a function.

Definition 5 (Adjoint function)**.** *For any function* $g:(0,\infty)\to(0,\infty)$ *the* adjoint function *of* g *is denoted by* g^* *and it is defined by*

$$g^*:(0,\infty)\to(0,\infty); \quad x\mapsto g^*(x):=\frac{1}{g\left(\frac{1}{x}\right)}.$$

If g is also continuous then we have

$$g^*\left(X^{-1}\right)=g(X)^{-1},\, g\left(X^{-1}\right)=g^*(X)^{-1} \qquad \left(X\in\mathscr{B}(\mathscr{H})^{++}\right). \tag{5}$$

The above computation rules are straightforward to check and will be used in the following computations several times.

The adjoint of a connection is defined as follows. Let σ be a connection and let g be the representing function of σ. The representing function g is a positive operator monotone function defined on the positive half-line, hence so is its adjoint, the function g^* (the only non-trivial fact which is used at this point is that the map $x\mapsto 1/x$ is operator monotone decreasing). That is, g^* is also a representing function of an operator connection, and this connection is said to be the adjoint of σ, and is denoted by σ^*.

The following claim shows that the adjoint of a connection can be expressed very explicitly. This fact is mentioned also in [6], see Eq. (4.1). The proof of this claim is rather straightforward, but we present it for the sake of completeness.

Claim 1. *Let $g:(0,\infty)\to(0,\infty)$ be an operator monotone function. Then we have*

$$M_{g^*}\left(A^{-1},B^{-1}\right)=\left(M_g(A,B)\right)^{-1}\qquad\left(A,B\in\mathscr{B}(\mathscr{H})^{++}\right). \tag{6}$$

Proof.

$$M_{g^*}\left(A^{-1},B^{-1}\right)=A^{-\frac{1}{2}}g^*\left(A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}\right)A^{-\frac{1}{2}}$$

$$=A^{-\frac{1}{2}}\left(g\left(\left(A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}}\right)^{-1}\right)\right)^{-1}A^{-\frac{1}{2}}=A^{-\frac{1}{2}}\left(g\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)\right)^{-1}A^{-\frac{1}{2}}$$

$$=\left(A^{\frac{1}{2}}g\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}}\right)^{-1}=\left(M_g(A,B)\right)^{-1}.$$

□

Remark 2. Let us note the following facts.

- The adjoint of the arithmetic mean is the harmonic mean, and vice versa.
- The geometric mean is self-adjoint.

The following claim shows that there is a one-to-one correspondence between the self-adjoint functions — that is, functions mapping $(0,\infty)$ into itself and satisfying $g(x)=\left(g\left(x^{-1}\right)\right)^{-1}$ — and the odd functions on the real line.

Claim 2. *Let $\varphi:\mathbb{R}\to\mathbb{R}$ be an odd function, that is $\phi(-t)=-\varphi(t)$, $(t\in\mathbb{R})$. Then the function*

$$g:(0,\infty)\to(0,\infty);\ x\mapsto g(x):=\exp\left(\varphi\left(\log(x)\right)\right)$$

is self-adjoint, that is, $g^*=g$.

Moreover, if a function $g:(0,\infty)\to(0,\infty)$ is self-adjoint, that is, it satisfies $g(x)=\left(g\left(x^{-1}\right)\right)^{-1}$, then the function $\varphi:\mathbb{R}\to\mathbb{R}$ defined by $\varphi(t):=\log\left(g\left(\exp(t)\right)\right)$ is odd.

This claim can be verified by straightforward computations. So, every odd function on the real line determines a self-adjoint function on $(0,\infty)$ and vice versa.

Remark 3. Note that every power function $f(x)=x^p$ $\left(p\in\mathbb{R}\right)$ is self-adjoint on $(0,\infty)$.

3. Main results

The following lemma is not difficult to prove, however, it leads to several interesting statements.

Lemma 3. *Let $f : (0,\infty) \to (0,\infty)$ be a continuous function and let g and h be positive operator monotone functions defined on $(0,\infty)$. Then the followings are equivalent.*

(a) *The function f is $\left(M_g, M_h\right)$-concave, that is,*

$$f\left(M_g\left(X,Y\right)\right) \geq M_h\left(f(X), f(Y)\right) \qquad \left(X, Y \in \mathscr{B}(\mathscr{H})^{++}\right). \tag{7}$$

(b) *The function f^* is $\left(M_{g^*}, M_{h^*}\right)$-convex, that is,*

$$f^*\left(M_{g^*}\left(X,Y\right)\right) \leq M_{h^*}\left(f^*(X), f^*(Y)\right) \qquad \left(X, Y \in \mathscr{B}(\mathscr{H})^{++}\right). \tag{8}$$

In particular, a continuous function $f : (0,\infty) \to (0,\infty)$ is concave with respect to the operator connection M_g if and only if the adjoint function f^* is convex with respect to the adjoint connection M_{g^*}.

Proof. Let us consider first the direction $(a) \Rightarrow (b)$. Let X and Y be arbitrary positive definite operators. By Claim 1 and the computation rules (5) we have

$$f^*\left(M_{g^*}\left(X,Y\right)\right) = f^*\left(\left(M_g\left(X^{-1}, Y^{-1}\right)\right)^{-1}\right) = \left(f\left(M_g\left(X^{-1}, Y^{-1}\right)\right)\right)^{-1}.$$

Taking the inverse reverses the semidefinite order, hence by the concavity assumption (7) we can majorize the latest term as follows:

$$\left(f\left(M_g\left(X^{-1}, Y^{-1}\right)\right)\right)^{-1} \leq \left(M_h\left(f\left(X^{-1}\right), f\left(Y^{-1}\right)\right)\right)^{-1}.$$

Taking Claim 1 into account again and using the computation rules (5) we get that

$$\begin{aligned}\left(M_h\left(f\left(X^{-1}\right), f\left(Y^{-1}\right)\right)\right)^{-1} &= M_{h^*}\left(\left(f\left(X^{-1}\right)\right)^{-1}, \left(f\left(Y^{-1}\right)\right)^{-1}\right) \\ &= M_{h^*}\left(f^*(X), f^*(Y)\right),\end{aligned}$$

and so the inequality (8) is verified.

The direction $(b) \Rightarrow (a)$ is very similar. Assuming (8) we can deduce that

$$\begin{aligned} f\left(M_g\left(X,Y\right)\right) &= f\left(\left(M_{g^*}\left(X^{-1}, Y^{-1}\right)\right)^{-1}\right) = \left(f^*\left(M_{g^*}\left(X^{-1}, Y^{-1}\right)\right)\right)^{-1} \\ &\geq \left(M_{h^*}\left(f^*\left(X^{-1}\right), f^*\left(Y^{-1}\right)\right)\right)^{-1} = M_h\left(\left(f^*\left(X^{-1}\right)\right)^{-1}, \left(f^*\left(Y^{-1}\right)\right)^{-1}\right) \\ &= M_h\left(f(X), f(Y)\right). \end{aligned}$$

□

Our main results (Theorem 4 and Theorem 5) are heavily based on this Lemma 3.

Theorem 4. *Any operator concave function is convex with respect to the harmonic mean, that is, if $f:(0,\infty)\mapsto(0,\infty)$ is operator concave, then*

$$f(X!Y)\leq f(X)!f(Y) \qquad \left(X,Y\in\mathscr{B}(\mathscr{H})^{++}\right),$$

where the symbol $!$ stands for the harmonic mean.

Proof. For any function $f:(0,\infty)\to(0,\infty)$ the followings are equivalent.

- f is operator monotone,
- f is operator concave,

see, e.g., [5, Thm. 4.43]. The adjoint of an operator monotone function is operator monotone (see the discussion after Definition 5 where the adjoint of a function was defined), so the adjoint of a positive operator concave function defined on the positive half-line is operator concave. That is, for an operator concave function $f:(0,\infty)\to(0,\infty)$ we have

$$f^{*}(X\nabla Y)\geq f^{*}(X)\nabla f^{*}(Y) \qquad \left(X,Y\in\mathscr{B}(\mathscr{H})^{++}\right), \tag{9}$$

where ∇ denotes the arithmetic mean. As mentioned in Remark 2, the harmonic mean is the adjoint of the arithmetic mean, hence by Lemma 3, the inequality (9) implies

$$f(X!Y)\leq f(X)!f(Y) \qquad \left(X,Y\in\mathscr{B}(\mathscr{H})^{++}\right),$$

which completes the proof. □

Example 3. The function $f(x)=x^p$ is operator concave, if $0\leq p\leq 1$, hence by Theorem 4, the inequality

$$(X!Y)^p\leq X^p!Y^p \qquad \left(p\in[0,1],\, X,Y\in\mathscr{B}(\mathscr{H})^{++}\right)$$

holds.

Theorem 5. *If $f:(0,\infty)\mapsto(0,\infty)$ is operator convex function such that the numerical function $f(x)$ is non-increasing, then f is concave with respect to the harmonic mean, that is,*

$$f(X!Y)\geq f(X)!f(Y) \qquad \left(X,Y\in\mathscr{B}(\mathscr{H})^{++}\right).$$

Proof. By the result of [2, Thm. 2.1 and Thm. 3.1], if $f:(0,\infty)\to(0,\infty)$ is a continuous function, then f is operator convex and numerically non-increasing if and only if f is operator monotone decreasing. The map $X\mapsto X^{-1}$ reverses the Löwner order on positive definite operators, so it is clear that if f is operator monotone decreasing, then so is f^*. Using again the result of [2, Thm. 2.1 and Thm. 3.1], this means that f^* is operator convex. And now, let us use Lemma 3 and the fact that the arithmetic mean is the adjoint of the harmonic mean (Remark 2) to deduce that f is concave with respect to the harmonic mean. □

Remark 4. Not only the numerically non-increasing operator convex functions are concave with respect to the harmonic mean. It is well-known that for any $p \in [1,2]$, the power function $f(x) = x^p$ is operator convex (see, e.g., [3, Chapter V]). As we have remarked before, every power function is self-adjoint (Remark 3). Using this self-adjoint property of the power functions and the fact that the arithmetic mean is the adjoint of the harmonic mean (Remark 2), we obtain by Lemma 3 that for any $1 \le p \le 2$ the function $f(x) = x^p$ is concave with respect to the harmonic mean, that is,

$$(X!Y)^p \ge X^p!Y^p \qquad \left(p \in [1,2],\, X, Y \in \mathscr{B}(\mathscr{H})^{++}\right).$$

So, we have seen that those operator convex functions which are non-increasing as numerical functions are concave with respect to the harmonic mean, and that there is a family of numerically increasing operator convex functions which are concave with respect to the harmonic mean, as well. Therefore, the following question naturally appears.

Problem 1. Is it true that every operator convex function is concave with respect to the harmonic mean?

4. Closing remarks and further open questions

Note that Lemma 3 has an easy but remarkable consequence for self-adjoint functions. Namely, that if f is a self-adjoint function (that is, $f = f^*$ holds), then f is concave with respect to the operator connection M_g if and only is f is convex with respect to the adjoint connection M_{g^*}. Recall that there is a lot of self-adjoint function in the sense that the self-adjoint functions are in a one-to-one correspondence with the odd functions on the real line (see Claim 2).

The situation is even more interesting if the connection M_g is also self-adjoint (and not just the function f.) In this case, by Lemma 3, if f is convex with respect to M_g, that is,

$$f\left(M_g(X,Y)\right) \le M_g\left(f(X), f(Y)\right) \qquad \left(X, Y \in \mathscr{B}(\mathscr{H})^{++}\right), \tag{10}$$

then f is also concave with respect to M_g, that is,

$$f\left(M_g(X,Y)\right) \ge M_g\left(f(X), f(Y)\right) \qquad \left(X, Y \in \mathscr{B}(\mathscr{H})^{++}\right). \tag{11}$$

The Löwner order on self-adjoint operators is antisymmetric, hence the inequalities (10) and (11) imply

$$f\left(M_g(X,Y)\right) = M_g\left(f(X), f(Y)\right) \qquad \left(X, Y \in \mathscr{B}(\mathscr{H})^{++}\right), \tag{12}$$

which means that f *preserves* the connection M_g. We summarize the result of the above argument in a proposition.

Proposition 6. *Let $f : (0, \infty) \to (0, \infty)$ be a continuous self-adjoint function and let M_g be a self-adjoint operator connection. Then the followings are equivalent.*

(i) *The function f is convex with respect to M_g, that is,* (10) *holds.*
(ii) *The function f is concave with respect to M_g, that is,* (11) *holds.*
(iii) *The function f preserves the connection M_g, that is,* (12) *holds.*

The geometric mean $A\#B = A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{1/2}A^{1/2}$ is an important example of self-adjoint connections. Proposition 6 shows that if a self-adjoint function (for example, a power function) is convex (or concave) with respect to the geometric mean, then is *preserves* the geometric mean. The reader who is interested in the description of the preserver transformations of the geometric mean and other operator means should consult the works [7] and [8] of Lajos Molnár.

We have seen above (Proposition 6) that every self-adjoint connection M_g has the property that if a self-adjoint function f is convex with respect to M_g, then f preserves the connection M_g (in the sense of Eq. (12)). It is quite natural to ask whether this property characterizes the self-adjoint connections or not. We formulate this question precisely as follows.

Problem 2. Assume that M_g is a operator connection such that if a self-adjoint function f is convex with respect to M_g then it necessarily preserves M_g. Is it true that M_g is a self-adjoint connection?

Similarly, it follows from Proposition 6 that every self-adjoint function f has the property that if f is convex with respect to a self-adjoint connection M_g, then f preserves M_g. It is also natural to ask whether this property characterizes the self-adjoint functions or not. We finish our paper by proposing this open question.

Problem 3. Assume that $f : (0, \infty) \to (0, \infty)$ is a continuous function such that for any self-adjoint connection M_g the following holds: if f is convex with respect to M_g then it preserves M_g. Is it true that f is a self-adjoint function?

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