1. Operations with binary vectors and communication over Binary Symmetric Channel (BSC)

Coding Technology

Problem 1

A sequence of random bits has independent bits. The probability of 1 is $P_b = 0.03$, and the probability of 0 is $1 - P_b = 0.97$.

(a) What is the probability of the sequence 01000100?

(b) What is the probability that the number of 1's in an 8-bit sequence is exactly 2?

(c) What is the probability that the number of 1's in an 8-bit sequence is 2 or higher?

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Solution.

(a)

$$\begin{array}{cccccccc} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 0.97 \cdot & 0.03 \cdot & 0.97 \cdot & 0.97 \cdot & 0.97 \cdot & 0.03 \cdot & 0.97 \cdot & 0.97 \end{array} = 0.03^2 \cdot 0.97^6 \approx 0.00075.$$

Problem 1

Solution.

(b) The number of 8-bit sequences that contain exactly 2 ones is $\binom{8}{2}$, and each such sequence has probability equal to part (a), so

$$P(2 \text{ ones in an 8-bit sequence}) = \binom{8}{2} \cdot 0.03^2 \cdot 0.97^6 \approx 0.0210.$$

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(c)

$$P(2 \text{ or more ones in an 8-bit sequence}) =$$
$$1 - 0.97^8 - \binom{8}{1} \cdot 0.03^1 \cdot 0.97^7 \approx 0.0223.$$

Problem 2

(a) For a BSC, the input vector is $u = (0010011)$ and the randomly generated error vector is $e = (1000001)$. The bit error probability is $P_b = 0.1$. What is the output vector of the channel?

(b) What is the probability of the error vector $e = (1000001)$?

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Solution.

(a) The output is the mod 2 sum of the input and the error vector:

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(b) The probability of the error vector is

$$P(e = 1000001) = 0.1^2 \cdot (1 - 0.1)^5 \approx 0.005905.$$

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(b) $v = u + e \implies e = u + v$

$$\begin{array}{ll} \text{input:} & 0010011 \\ \text{output:} & 1010010 \\ \text{error:} & 1000001 \end{array}$$

Problem 4

For a BSC, the input vector is $u = (00100111)$ output vector is $v = (10100101)$.

(a) What is the error vector?

(b) The channel error probability is $P_b = 0.01$. What is the conditional probability that the output vector is the above v, assuming that the input vector is the above u?

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Solution.

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$$e = u + v = (10000010)$$

(b)

$$P(v = (10100101) | u = (00100111)) =$$
$$P_b^2(1 - P_b)^6 = 0.01^2 \cdot 0.99^6 \approx 0.00009415.$$

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For a BSC, the bit error probability is $P_b = 0.2$. Calculate how many errors we need to correct if the desired block error probability (QoS) is less than 0.001, and the block length is $n = 30$.

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- i is the number of incorrect bits in a block;
- $30 - i$ is the number of correct bits;
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$$\left.\begin{array}{l} \sum_{i=13}^{30} \binom{30}{i} 0.2^i 0.8^{30-i} \approx 0.00311 \\ \sum_{i=14}^{30} \binom{30}{i} 0.2^i 0.8^{30-i} \approx 0.000902 \end{array}\right\} \implies t = 13.$$

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(a) What is the number of 5-dimensional binary vectors with Hamming distance 3 from the vector (01010)?

(b) What is the number of binary vectors inside the sphere with radius 3 with center (01010)?

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Solution.

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(b)

$$\sum_{i=0}^{3} \binom{5}{i} = \binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} =$$
$$1 + 5 + 10 + 10 = 26.$$

Problem 7

Calculate the weight of the vector (000100011000111101000).

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Solution. The vector contains 8 ones, so

$$w(000100011000111101000) = 8.$$

Problem 8

Calculate the following matrix-vector multiplication according to mod 2 arithmetics.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Problem 8

Solution.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

(Add columns 2, 3 and 7 of the matrix componentwise.)

Problem 9

Calculate the following matrix-vector multiplication according to mod 2 arithmetics.

$$(1001) \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Problem 9

Solution.

$$(1001) \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} =$$

$$(01001110).$$

(Add rows 1 and 4 of the matrix componentwise.)