1. Operations with binary vectors and communication over Binary Symmetric Channel (BSC)

Coding Technology
Problem 1

(a) For a BSC, the input vector is $u = (0010011)$ and the randomly generated error vector is $e = (1000001)$. The bit error probability is $P_b = 0.1$. What is the output vector of the channel?

(b) What is the probability of the error vector $e = (1000001)$?
Problem 1

(a) For a BSC, the input vector is $u = (0010011)$ and the randomly generated error vector is $e = (1000001)$. The bit error probability is $P_b = 0.1$. What is the output vector of the channel?

(b) What is the probability of the error vector $e = (1000001)$?

Solution.

(a) The output is the mod 2 sum of the input and the error vector:

\[
\begin{array}{c}
0010011 \\
+ 1000001 \\
\hline
1010010
\end{array}
\]
Problem 1

(a) For a BSC, the input vector is \( u = (0010011) \) and the randomly generated error vector is \( e = (1000001) \). The bit error probability is \( P_b = 0.1 \). What is the output vector of the channel?

(b) What is the probability of the error vector \( e = (1000001) \)?

Solution.

(a) The output is the mod 2 sum of the input and the error vector:

\[
\begin{array}{c}
0010011 \\
+ 1000001 \\
\hline
1010010
\end{array}
\]

(b) The probability of the error vector is

\[
P(e = 1000001) = 0.1^2 \cdot (1 - 0.1)^5 \approx 0.005905.
\]
Problem 2

(a) \( u = (0010011) \) is the input and \( v = (1010010) \) is the corresponding output of a BSC. What is the Hamming distance between \( u \) and \( v \)?

(b) What is the error vector generated by the BSC?
Problem 2

(a) $u = (0010011)$ is the input and $v = (1010010)$ is the corresponding output of a BSC. What is the Hamming distance between $u$ and $v$?

(b) What is the error vector generated by the BSC?

Solution.

(a) 

<table>
<thead>
<tr>
<th>Input</th>
<th>0010011</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output</td>
<td>1010010</td>
</tr>
</tbody>
</table>

$\downarrow \quad \downarrow$

$1 + 1 = 2$
Problem 2

(a) $u = (0010011)$ is the input and $v = (1010010)$ is the corresponding output of a BSC. What is the Hamming distance between $u$ and $v$?

(b) What is the error vector generated by the BSC?

Solution.

(a)

$\begin{align*}
\text{input:} & \quad 0010011 \\
\text{output:} & \quad 1010010 \\
\downarrow & \quad \downarrow \\
1 & + 1 = 2
\end{align*}$

(b) $v = u + e \implies e = u + v$

$\begin{align*}
\text{input:} & \quad 0010011 \\
\text{output:} & \quad 1010010 \\
\text{error:} & \quad 1000001
\end{align*}$
Problem 3

(a) What is the error vector if the input vector is \( u = (00100111) \) output vector is \( v = (10100101) \)?

(b) If the channel error probability is \( P_b = 0.01 \), what is the conditional probability that the output vector is \( v = (10100101) \), assuming that the input vector is \( u = (00100111) \)?
Problem 3

(a) What is the error vector if the input vector is $u = (00100111)$ output vector is $v = (10100101)$?

(b) If the channel error probability is $P_b = 0.01$, what is the conditional probability that the output vector is $v = (10100101)$, assuming that the input vector is $u = (00100111)$?

Solution.

(a) The error vector is input + output mod 2:

$$e = u + v = (10000010)$$
Problem 3

(a) What is the error vector if the input vector is $u = (00100111)$ and the output vector is $v = (10100101)$?

(b) If the channel error probability is $P_b = 0.01$, what is the conditional probability that the output vector is $v = (10100101)$, assuming that the input vector is $u = (00100111)$?

Solution.

(a) The error vector is input + output mod 2:

$$e = u + v = (10000010)$$

(b)

$$P(v = (10100101)|u = (00100111)) = P_b^2(1 - P_b)^6 = 0.01^2 \cdot 0.99^6 \approx 0.00009415.$$
Problem 4

For a BSC, the bit error probability is $P_b = 0.2$. Calculate how many errors we need to correct if the desired block error probability (QoS) is less than 0.001, and the block length is $n = 30$. 

Solution. We use the following notation.
- $i$ is the number of incorrect bits in a block;
- $30 - i$ is the number of correct bits;
- $t$ is the number of errors we can correct.

We need to find the smallest $t$ for which

$$
\sum_{i=0}^{30} i = t + 1 \left(30^i\right) 0.2^i 0.8^{30-i} \leq 0.001.
$$

$$
\sum_{i=0}^{13} i = 13 \left(30^i\right) 0.2^i 0.8^{30-i} \approx 0.00311
$$

$$
\sum_{i=0}^{14} i = 14 \left(30^i\right) 0.2^i 0.8^{30-i} \approx 0.000902
$$

$$
\Rightarrow t = 13.
$$
Problem 4

For a BSC, the bit error probability is $P_b = 0.2$. Calculate how many errors we need to correct if the desired block error probability (QoS) is less than 0.001, and the block length is $n = 30$.

Solution. We use the following notation.

- $i$ is the number of incorrect bits in a block;
- $30 - i$ is the number of correct bits;
- $t$ is the number of errors we can correct.

We need to find the smallest $t$ for which

$$30 \sum_i = t + 1 \left(30 - 0.2i\right) \approx 0.001.$$
Problem 4

For a BSC, the bit error probability is $P_b = 0.2$. Calculate how many errors we need to correct if the desired block error probability (QoS) is less than 0.001, and the block length is $n = 30$.

Solution. We use the following notation.

- $i$ is the number of incorrect bits in a block;
- $30 - i$ is the number of correct bits;
- $t$ is the number of errors we can correct.

We need to find the smallest $t$ for which

$$\sum_{i=t+1}^{30} \binom{30}{i} 0.2^i 0.8^{30-i} \leq 0.001.$$
Problem 4

For a BSC, the bit error probability is $P_b = 0.2$. Calculate how many errors we need to correct if the desired block error probability (QoS) is less than 0.001, and the block length is $n = 30$.

Solution. We use the following notation.

- $i$ is the number of incorrect bits in a block;
- $30 - i$ is the number of correct bits;
- $t$ is the number of errors we can correct.

We need to find the smallest $t$ for which

$$
\sum_{i=t+1}^{30} \binom{30}{i} 0.2^i 0.8^{30-i} \leq 0.001.
$$

\[
\begin{align*}
\sum_{i=13}^{30} \binom{30}{i} 0.2^i 0.8^{30-i} &\approx 0.00311 \\
\sum_{i=14}^{30} \binom{30}{i} 0.2^i 0.8^{30-i} &\approx 0.000902
\end{align*}
\]

\[
\left\{ \begin{array}{c}
\sum_{i=13}^{30} \binom{30}{i} 0.2^i 0.8^{30-i} \approx 0.00311 \\
\sum_{i=14}^{30} \binom{30}{i} 0.2^i 0.8^{30-i} \approx 0.000902
\end{array} \right\} \implies t = 13.
\]
Problem 5

(a) What is the number of 5-dimensional binary vectors with Hamming distance 3 from the vector (01010)?

(b) What is the number of binary vectors inside the sphere with radius 3 with center (01010)?
Problem 5

(a) What is the number of 5-dimensional binary vectors with Hamming distance 3 from the vector $(01010)$?

(b) What is the number of binary vectors inside the sphere with radius 3 with center $(01010)$?

Solution.

(a) \[ \binom{5}{3} = \frac{5!}{3! \cdot 2!} = 10. \]
Problem 5

(a) What is the number of 5-dimensional binary vectors with Hamming distance 3 from the vector (01010)?

(b) What is the number of binary vectors inside the sphere with radius 3 with center (01010)?

Solution.

(a)

\[ \binom{5}{3} = \frac{5!}{3! \cdot 2!} = 10. \]

(b)

\[ \sum_{i=0}^{3} \binom{5}{i} = \binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3} = 1 + 5 + 10 + 10 = 26. \]
Calculate the weight of the vector \((000100011000111101000)\).
Problem 6

Calculate the weight of the vector \((000100011000111101000)\).

Solution. The vector contains 8 ones, so

\[ w(000100011000111101000) = 8. \]
Problem 7

Calculate the following matrix-vector multiplication according to mod 2 arithmetics.

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
\end{bmatrix}
\cdot
\begin{bmatrix}
0 \\
1 \\
1 \\
0 \\
0 \\
0 \\
1 \\
0 \\
\end{bmatrix}
\]
Problem 7

Solution.

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
0 \\
1 \\
1 \\
0 \\
0 \\
0 \\
1 \\
0
\end{bmatrix}
= \begin{bmatrix}
0 \\
1 \\
1 \\
0 \\
0 \\
0 \\
1 \\
0
\end{bmatrix}.
\]

(Add columns 2, 3 and 7 of the matrix componentwise.)
Problem 8

Calculate the following matrix-vector multiplication according to mod 2 arithmetics.

\[(1001) \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}\]
Solution.

\[(1001) \cdot \begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 & 1
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0
\end{bmatrix} \]

(01001110).

(Add rows 1 and 4 of the matrix componentwise.)