Practice problems for the midterm test

Coding Technology

Problem 1

A binary error correction code has codewords

$$000010, \qquad 110011, \qquad 101001, \qquad 111100.$$

(a) What are the parameters of the code?

(b) Compute the error detection and error correction capabilities of the code.

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(b) Based on pairwise comparison, the minimal Hamming-distance among codewords is $d_{\min} = 3$:

000010	000010	000010	110001	110011	101001
110011	101001	111100	101011	111100	111100

The code can detect $d_{\min} - 1 = 2$ errors and correct $\lfloor \frac{d_{\min}-1}{2} \rfloor = 1$ error.

Problem 2

A binary linear error-correcting code has parity check matrix

$$H = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Is the code systematic?

(b) Determine the generator matrix G.

(c) Is this a Hamming-code?

(d) List all codewords.

(e) How many errors can the code correct?

(f) Compute the syndrome vector and the error group for the error vector $e = (10100)$. What is the group leader?

Problem 2

Solution.

(a) The code is systematic, because the rightmost 3×3 block of H is the identity matrix:

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(b) For systematic linear codes,

$$G = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix},$$

where the leftmost 2×2 block of G is the identity matrix, and the rightmost 2×3 block is the transpose of the leftmost 3×2 block of H.

Problem 2

(c) For Hamming codes, $n+1=2^{n-k}$ needs to hold, but based on the size of G, $n=5$ and $k=2$, for which $n+1=2^{n-k}$ does not hold: $n+1=6\neq 8=2^{n-k}$. This is not a Hamming code.

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(d) For linear codes, the codewords are all linear combinations of the rows of G:

$$(00000), \qquad (10101), \qquad (01111), \qquad (11010).$$

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(e) Since this is a linear code,

$$d_{\text{min}} = \min_{0 \neq c \text{ codeword}} w(c) = 3,$$

and the code can correct $\lfloor \frac{3-1}{2} \rfloor = 1$ error.

Problem 2

(f) The syndrome vector corresponding to the error vector $e = (10100)$ is

$$s^T = He^T = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

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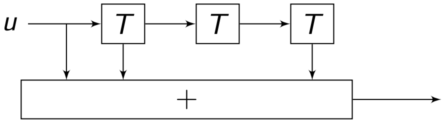
The error group can be obtained by adding each codeword to e:

$$\{(10100), (00001), (11011), (01100)\}.$$

The group leader is the vector with minimal weight: (00001).

Problem 3

Consider GF(8) with the irreducible polynomial $p(y) = y^3 + y + 1$. The following shift register architecture defines a linear code over the GF(8):



(a) What is the generator polynomial $g(x)$ of the code?

(b) What are the parameters of the code?

(c) What are the error correction capabilities of the code?

Problem 3

Solution.

(a) The given shift register architecture is multiplication by a polynomial. It contains no constant factors, so all coefficients of the polynomial are either 0 or 1, depending on whether the corresponding coefficient is included in the sum or not. Altogether, the architecture is multiplication by the polynomial $g(x) = x^3 + x + 1$m, so the generating polynomial of the code is $x^3 + x + 1$.

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(b) Over GF(8), generator polynomials with 0-1 coefficients are typical for BCH codes. Let's check whether this polynomial is the generator polynomial of a BCH code.

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(b) Over GF(8), generator polynomials with 0-1 coefficients are typical for BCH codes. Let's check whether this polynomial is the generator polynomial of a BCH code.

The conjugate groups and minimal polynomials over GF(8) are

$$\{1\} \to x - 1$$
$$\{y, y^2, y^4\} \to x^3 + x + 1$$
$$\{y^3, y^5, y^6\} \to x^3 + x^2 + 1$$

Problem 3

Solution.

(b) We need to check whether the generator polynomial is a product of some of the minimal polynomials Actually, $g(x)$ is equal to the minimal polynomial of the group $\{y, y^2, y^4\}$, so yes.

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So this is a BCH code; the parameters are $n = 8 - 1 = 7$ and the degree of the generator polynomial is $n - k = 3$, so $k = 4$, and this is a C(7,4) code.

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(c) The roots of $g(x)$ contain y^1 and y^2 (along with their entire conjugate group), but not y^3, so this code can correct $t = 1$ error.