Basic Probability 1

Stochastics

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1 Probability space, events

2 Conditional probability

3 Random variables, distributions
A probability space is an \((\Omega, \mathbb{P}, \mathcal{F})\) triple, where

- \(\Omega\) is the set of outcomes,
- \(\mathcal{F}\) is the set of events, which are subsets of \(\Omega\), and
- \(\mathbb{P}\) is the probability function, defined on \(\mathcal{F}\).
Example. We flip a fair coin 3 times. Then

$$\Omega = \{HHH, HHT, HTH, THH, THT, TTH, TTT\}$$

where H denotes heads, T denotes tails.
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\[ \Omega = \{HHH, HHT, HTT, HTH, THH, THT, TTH, TTT\} \]

where H denotes heads, T denotes tails. Example events:

- \( A = \text{“the first flip is heads”} = \{HHH, HHT, HTH, HTT\}, \)
- \( B = \text{“all three flips are the same”} = \{HHH, TTT\}. \)
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where H denotes heads, T denotes tails. Example events:

\[ A = "the first flip is heads" = \{HHH, HHT, HTH, HTT\}, \]

\[ B = "all three flips are the same" = \{HHH, TTT\}. \]

The probability function is such that

\[ \mathbb{P}(HHH) = \cdots = \mathbb{P}(TTT) = \frac{1}{8}, \]

while

\[ \mathbb{P}(A) = \frac{2}{8} \quad \text{and} \quad \mathbb{P}(B) = \frac{4}{8}. \]
Properties of the probability function

The probability function always satisfies the following:

\[ P(\emptyset) = 0, \]
\[ P(\Omega) = 1, \]

and if \( A_1, A_2, \ldots \) are disjoint events, then

\[ P\left( \bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} P(A_i). \]

(This property is called \( \sigma \)-additivity.)
Example. We select a random point from the interval $[0, 1]$ uniformly. Then

$$\Omega = [0, 1],$$

and

$$\mathbb{P}(\{0.5\}) = 0,$$

$$\mathbb{P}([0.5, 0.7]) = 0.2,$$

$$\mathbb{P}([0.5, 0.7] \cup [0.9, 1]) = 0.3.$$
For $A$ and $B$ events, the *conditional probability of $A$ assuming $B$* is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$ 

$A$ and $B$ are *independent* if

$$P(A \cap B) = P(A)P(B),$$

or, equivalently,

$$P(A|B) = P(A).$$

Independence is symmetric.
Example. We flip a fair coin three times. What is the conditional probability that the first flip is heads, assuming there are at least 2 heads?

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and so

\[ A \cap B = \{HHH, HHT, HTH\}. \]

Then

\[ \mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \frac{3/8}{4/8} = \frac{3}{4}. \]
Conditional probability

Theorem (Bayes)

\[ P(B|A) = \frac{P(A|B)P(B)}{P(A)} . \]
Conditional probability

**Theorem (Bayes)**

\[ P(B|A) = \frac{P(A|B)P(B)}{P(A)} . \]

\( B_1, \ldots, B_k \) are a complete system of events if

\[ B_i \cap B_j = \emptyset \text{ for } i \neq j, \]

\[ B_1 \cup \cdots \cup B_k = \Omega. \]

**Theorem (Total probability)**

*If \( B_1, \ldots, B_k \) is a complete set of events, then*

\[ P(A) = P(A|B_1)P(B_1) + \cdots + P(A|B_k)P(B_k). \]
Informally, a random variable is a random number.

Formally, for a given probability space, a \textit{random variable} $X$ is an $\Omega \to \mathbb{R}$ function.
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Formally, for a given probability space, a random variable $X$ is an $\Omega \to \mathbb{R}$ function.

Example. We flip a fair coin three times; $X$ is the number of heads. Then

\[
X(\text{HHH}) = 3, \\
X(\text{HHT}) = X(\text{HTH}) = X(\text{THH}) = 2, \\
X(\text{HTT}) = X(\text{THT}) = X(\text{TTH}) = 1, \\
X(\text{TTT}) = 0.
\]
If $X$ can only take nonnegative integer values, we say $X$ is a \textit{discrete} random variable, and if $X$ can take any real value, then we say $X$ is a \textit{continuous} random variable.
Random variables

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Random variables

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Example. I go to the street and start counting cars.

Let $T$ denote the time when the first car arrives. Then $T$ is continuous.
If $X$ can only take nonnegative integer values, we say $X$ is a *discrete* random variable, and if $X$ can take any real value, then we say $X$ is a *continuous* random variable.

Example. I go to the street and start counting cars.

Let $T$ denote the time when the first car arrives. Then $T$ is continuous.

Let $X$ denote the number of cars in 2 minutes. Then $X$ is discrete.
Distributions

The distribution of a random variable is, roughly speaking, what values can it take and with what probability. If $X$ is discrete, then the distribution of $X$ can be described by the values

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P(X = k) = p_k, \quad k = 0, 1, \ldots
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Distributions

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Example. We flip a fair coin three times; $X$ is the number of heads. Then

$$
P(X = 0) = \frac{1}{8}, \quad P(X = 1) = \frac{3}{8},
$$

$$
P(X = 2) = \frac{3}{8}, \quad P(X = 3) = \frac{1}{8}.
$$
For $X$ continuous, the same description for the distribution does not work. For example, if $X$ denotes the time we need to wait for the next car on the street to pass, the probability that $X$ is exactly 1 minute is 0.
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Accordingly, for a continuous random variable $X$, its distribution is described by the *cumulative distribution function* (cdf)

$$F(x) = \mathbb{P}(X < x)$$

where $x$ is a real variable.
Properties of cdf’s:

- $F(x)$ is increasing,
- $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to \infty} F(x) = 1$.

Any function with the above properties is a valid cdf.
If $F$ is differentiable, then $f(x) = \frac{dF(x)}{dx}$ is the corresponding probability density function (pdf). Properties of pdf’s:

- $f(x)$ is nonnegative,
- $\int_{-\infty}^{\infty} f(x)dx = 1$.

Any function with the above properties is a valid pdf.
For a given set $A \subseteq \mathbb{R}$,

\[
P(X \in A) = \begin{cases} 
\sum_{k \in A} p_k & \text{for } X \text{ discrete} \\
\int_{A} f(x) \, dx & \text{for } X \text{ continuous}
\end{cases}
\]
Distributions

For a given set $A \subseteq \mathbb{R}$,

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\end{cases}$$

A discrete random variable is more likely to take values $k$ for which $p_k$ is higher.
A continuous random variable is more likely to take values near $x$ where $f(x)$ is higher.
The *mean* or *expectation* or *expected value* of a random variable is

\[
\mathbb{E}(X) = \begin{cases} 
\sum_{k=0}^{\infty} kp_k & \text{for } X \text{ discrete} \\
\int_{\mathbb{R}} xf(x)\,dx & \text{for } X \text{ continuous}
\end{cases}
\]

Roughly speaking, the mean describes the average value of $X$.

Example. If $X$ is a roll with a fair 6-sided die, then

\[
\mathbb{E}(X) = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \frac{1}{6} \cdot 4 + \frac{1}{6} \cdot 5 + \frac{1}{6} \cdot 6 = \frac{7}{2}.
\]
Expectation of functions of $X$

If $X$ is a random variable and $g$ is an $\mathbb{R} \rightarrow \mathbb{R}$ function, then

$$
\mathbb{E}(g(X)) = \begin{cases} 
\sum_{k=0}^{\infty} g(k)p_k & \text{for } X \text{ discrete} \\
\int_{\mathbb{R}} g(x)f(x)dx & \text{for } X \text{ continuous}
\end{cases}
$$

Example. If $X$ is a roll with a fair 6-sided die, then

$$
\mathbb{E}(X^2) = \frac{1}{6} \cdot 1^2 + \frac{1}{6} \cdot 2^2 + \frac{1}{6} \cdot 3^2 + \frac{1}{6} \cdot 4^2 + \frac{1}{6} \cdot 5^2 + \frac{1}{6} \cdot 6^2 = \frac{91}{6}.
$$
Example. Let $X$ be a random variable such that $P(X = 49) = P(X = 51) = 1/2$.

Then $E(X) = \frac{1}{2} \cdot 49 + \frac{1}{2} \cdot 51 = 50$.

Similarly, let $Y$ be a random variable such that $P(Y = 0) = P(Y = 100) = 1/2$.

Then $E(Y) = \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 100 = 50 = E(X)$.

So $E(X)$ describes the average value of a random variable $X$, but not how far typically $X$ is from the mean.
The variance of $X$ is

$$\text{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2,$$

and the deviation of $X$ is

$$\mathbb{D}(X) = \sqrt{\text{Var}(X)} = \sqrt{\mathbb{E}(X^2) - (\mathbb{E}(X))^2}.$$

The deviation (and the variance) describe how close typically $X$ is to $\mathbb{E}(X)$.

Properties:

- $\text{Var}(X) = \mathbb{E}((X - \mathbb{E}(X))^2)$
- $\text{Var}(X) \geq 0$, and 0 only when $X$ is constant with probability 1.
Example. If $X$ is a roll with a fair 6-sided die, then we already know

$$E(X) = \frac{7}{2}, \quad E(X^2) = \frac{91}{6},$$

so

$$\text{Var}(X) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12},$$

and

$$\mathbb{D}(X) = \sqrt{\frac{35}{12}} \approx 1.71.$$
Properties of expectation and variance

Expectation is linear: for $a, b, c \in \mathbb{R}$ constants,

$$E(ax + by + c) = aE(X) + bE(Y) + c.$$
Properties of expectation and variance

Expectation is linear: for $a, b, c \in \mathbb{R}$ constants,

$$
\mathbb{E}(aX + bY + c) = a\mathbb{E}(X) + b\mathbb{E}(Y) + c.
$$

For $a, b \in \mathbb{R}$ constants,

$$
\mathbb{D}(aX + b) = a\mathbb{D}(X).
$$
The *conditional distribution* of a discrete random variable $X$ assuming an event $A$ can be given by

$$
P(X = k | A) = \frac{P(X = k, A)}{P(A)},$$

and the conditional cdf of a continuous random variable $X$ assuming an event $A$ can be given by

$$F(x | A) = P(X \leq x | A) = \frac{P(X \leq x, A)}{P(A)},$$

and the conditional pdf is $f(x | A) = \frac{dF(x | A)}{dx}$.
The conditional expectation of $X$ assuming an event $A$ is

$$
\mathbb{E}(X|A) = \begin{cases} 
\sum_{k=0}^{\infty} k \mathbb{P}(X = k|A) & \text{for } X \text{ discrete} \\
\int_{\mathbb{R}} xf(x|A)dx & \text{for } X \text{ continuous}
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**Theorem (Total expectation)**

If $B_1, \ldots, B_k$ is a complete set of events, then

\[
\mathbb{E}(X) = \mathbb{E}(X|B_1)\mathbb{P}(B_1) + \cdots + \mathbb{E}(X|B_k)\mathbb{P}(B_k).
\]
Problem 1

We roll two fair dice. Let $A$ denote the event that the sum of the two rolls are 6. Let $B$ denote the event that the first roll is 4. Show that $A$ and $B$ are not independent. Let $C$ denote the event that the sum of the two rolls is 7. Is $B$ independent from $C$?
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Solution.

$$\Omega = \{11, 12, \ldots, 66\},$$

$$P(11) = P(12) = \cdots = P(66) = \frac{1}{36}.$$
Problem 1

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Solution.

\[ \Omega = \{11, 12, \ldots, 66\}, \]
\[ \mathbb{P}(11) = \mathbb{P}(12) = \cdots = \mathbb{P}(66) = \frac{1}{36}. \]

\[ A = \{15, 24, 33, 42, 51\}, \quad \mathbb{P}(A) = \frac{5}{36}. \]

\[ B = \{41, 42, 43, 44, 45, 46\}, \quad \mathbb{P}(B) = \frac{6}{36}. \]
Problem 1

\[ A \cap B = \{42\}, \quad \mathbb{P}(A) = \frac{1}{36}, \]

so

\[ \mathbb{P}(A \cap B) = \frac{1}{36} \neq \mathbb{P}(A)\mathbb{P}(B) = \frac{5}{36} \cdot \frac{1}{36} = \frac{5}{216} \]

and \( A \) and \( B \) are not independent.
Problem 1

\[ A \cap B = \{42\}, \quad \mathbb{P}(A) = \frac{1}{36}, \]

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and \( A \) and \( B \) are not independent.

Next,
\[ C = \{16, 25, 34, 43, 52, 61\}, \quad \mathbb{P}(C) = \frac{6}{36}, \]
\[ B \cap C = \{43\}, \quad \mathbb{P}(B \cap C) = \frac{1}{36}. \]

Now
\[ \mathbb{P}(B \cap C) = \frac{1}{36} = \mathbb{P}(B) \mathbb{P}(C) = \frac{6}{36} \cdot \frac{6}{36} = \frac{1}{36} \]
Problem 3

Dennis has 2 identically-looking dice, one of which is fair (it gives the numbers 1, 2, 3, 4, 5 and 6 with probability $\frac{1}{6} - \frac{1}{6}$ each), but the other one is loaded: 6 has a probability of $\frac{1}{2}$. Dennis picks one of them at random and rolls it twice. What is the probability that he rolls two sixes? What is the conditional probability of the event that he picked the loaded die, assuming he rolls two sixes?
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Solution. Define the events

\[ A = \{ \text{he rolls two sixes} \}, \]
\[ B_1 = \{ \text{he picks the fair die} \}, \]
\[ B_2 = \{ \text{he picks the loaded die} \}. \]
Problem 3

The information given is the following:

\[ P(B_1) = P(B_2) \]

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The information given is the following:

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\[ \mathbb{P}(A|B_1) = \left( \frac{1}{6} \right)^2, \]

\[ \mathbb{P}(A|B_2) = \left( \frac{1}{2} \right)^2. \]
Problem 3

The information given is the following:

\[ P(B_1) = P(B_2) \]

because he picks at random, and

\[ P(A|B_1) = \left( \frac{1}{6} \right)^2, \]

\[ P(A|B_2) = \left( \frac{1}{2} \right)^2. \]

Then we can use total probability to compute

\[ P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) = \frac{1}{2} \cdot \frac{1}{36} + \frac{1}{2} \cdot \frac{1}{4} = \frac{10}{72}. \]
Finally, to compute $\mathbb{P}(B_2|A)$ we can use Bayes:

$$\mathbb{P}(B_2|A) = \frac{\mathbb{P}(A|B_2)\mathbb{P}(B_2)}{\mathbb{P}(A)} = \frac{1}{2} \cdot \frac{1}{4} = \frac{9}{10},$$

so he picked the loaded die with 90% probability.
Problem 6

A miner gets lost in the mine. He is in a chamber with 5 doors. Door 1 leads to a tunnel to the exit after 2 hours of walking. Door 2 leads to a tunnel to Door 3 after 1 hours of walking. Door 4 leads to a tunnel to Door 5 after 3 hours of walking. The miner picks a door at random, goes through the tunnel, but whenever he gets back to the chamber, he forgets his previous choices, and picks one of the five doors at random again.

$X$ denotes the total time it takes him to get to the exit. Calculate $E(X)$. (Hint: let $B_2$ denote the event that he picks Door 2 first. Argue that $E(X|B_2) = E(X) + 1$. Use total probability.)
Problem 6

Solution.

If he picks door 2, he will walk 1 hour, then get back to the chamber. From that, the time he has to walk has the same distribution as originally, in addition to the 1 hour he already spent, so $E(X|B_2) = E(X) + 1$. 
If he picks door 2, he will walk 1 hour, then get back to the chamber. From that, the time he has to walk has the same distribution as originally, in addition to the 1 hour he already spent, so

\[ \mathbb{E}(X | B_2) = \mathbb{E}(X) + 1. \]
Similarly for the other doors, we have

\[
\mathbb{E}(X|B_1) = 2, \\
\mathbb{E}(X|B_2) = \mathbb{E}(X|B_3) = \mathbb{E}(X) + 1, \\
\mathbb{E}(X|B_4) = \mathbb{E}(X|B_5) = \mathbb{E}(X) + 3.
\]
Problem 6

Similarly for the other doors, we have

\[ E(X|B_1) = 2, \]
\[ E(X|B_2) = E(X|B_3) = E(X) + 1, \]
\[ E(X|B_4) = E(X|B_5) = E(X) + 3. \]

Applying total expectation, we have

\[
E(X) = E(X|B_1)P(B_1) + E(X|B_2)P(B_2) + \\
E(X|B_3)P(B_3) + E(X|B_4)P(B_4) + E(X|B_5)P(B_5) = \\
2 \cdot \frac{1}{5} + (E(X) + 1) \cdot \frac{1}{5} + (E(X) + 1) \cdot \frac{1}{5} + \\
(E(X) + 3) \cdot \frac{1}{5} + (E(X) + 3) \cdot \frac{1}{5} = \frac{4}{5}E(X) + 2,
\]
Problem 6

Similarly for the other doors, we have

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\[ \mathbb{E}(X|B_2) = \mathbb{E}(X|B_3) = \mathbb{E}(X) + 1, \]
\[ \mathbb{E}(X|B_4) = \mathbb{E}(X|B_5) = \mathbb{E}(X) + 3. \]

Applying total expectation, we have

\[
\mathbb{E}(X) = \mathbb{E}(X|B_1)\mathbb{P}(B_1) + \mathbb{E}(X|B_2)\mathbb{P}(B_2) + \mathbb{E}(X|B_3)\mathbb{P}(B_3) + \mathbb{E}(X|B_4)\mathbb{P}(B_4) + \mathbb{E}(X|B_5)\mathbb{P}(B_5) = 2 \cdot \frac{1}{5} + (\mathbb{E}(X) + 1) \cdot \frac{1}{5} + (\mathbb{E}(X) + 1) \cdot \frac{1}{5} + 
\]
\[
(\mathbb{E}(X) + 3) \cdot \frac{1}{5} + (\mathbb{E}(X) + 3) \cdot \frac{1}{5} = \frac{4}{5} \mathbb{E}(X) + 2,
\]

whose solution is \( \mathbb{E}(X) = 10. \)
Problem 7

At a sports competition, participants have to throw a ball as far as possible. Let $X$ denote the result of a single throw of Jane (in meters). $X$ has the following probability density function:

$$f(x) = \begin{cases} \frac{75}{x^2} & 30 \leq x \leq 50 \\ 0 & \text{otherwise} \end{cases}$$

(a) Calculate the probability that Jane throws further away than 45 meters.
(b) Calculate the cumulative distribution function of $X$.
(c) Calculate $E[X]$.
(d) Each participant can throw the ball 3 times, and their score is the maximum of the 3 throws. Calculate the distribution of Jane’s score (we assume that different throws are independent).
Solution.

(a) 

\[ P(X > 45) = \int_{45}^{50} \frac{75}{x^2} \, dx = \left[ -\frac{75}{x} \right]_{x=45}^{50} = \frac{75}{45} - \frac{75}{50} = \frac{1}{6}. \]
Problem 7

Solution.

(a)
\[ P(X > 45) = \int_{45}^{50} \frac{75}{x^2} \, dx = \left[ -\frac{75}{x} \right]_{x=45}^{50} = \frac{75}{45} - \frac{75}{50} = \frac{1}{6}. \]

(b)
\[ \int_{30}^{x} \frac{75}{y^2} \, dy = \frac{75}{30} - \frac{75}{x}, \]
Problem 7

Solution.

(a) 
\[ P(X > 45) = \int_{45}^{50} \frac{75}{x^2} \, dx = \left[ -\frac{75}{x} \right]_{x=45}^{50} = \frac{75}{45} - \frac{75}{50} = \frac{1}{6}. \]

(b) 
\[ \int_{30}^{x} \frac{75}{y^2} \, dy = \frac{75}{30} - \frac{75}{x}, \]
and so 
\[ P(X < x) = \begin{cases} 
0 & x \leq 30 \\
\frac{75}{30} - \frac{75}{x} & 30 < x \leq 50 \\
1 & x > 50.
\end{cases} \]
Problem 7

(c)

\[ E(X) = \int_{\mathbb{R}} x \cdot f(x) \, dx = \int_{30}^{50} x \cdot \frac{75}{x^2} \, dx = \]
\[ [75 \log(x)]_{x=30}^{50} = 75 \log(50) - 75 \log(30) \approx 38.1. \]
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(c) 

\[ E(X) = \int \limits_{\mathbb{R}} xf(x)\,dx = \int_{30}^{50} x \cdot \frac{75}{x^2} \,dx = \left[ 75 \log(x) \right]_{x=30}^{50} = 75 \log(50) - 75 \log(30) \approx 38.1. \]

Let \( X_1, X_2, X_3 \) denote the three throws, and let 

\[ Y = \max(X_1, X_2, X_3). \]
(c) \[
\mathbb{E}(X) = \int_{\mathbb{R}} xf(x)\,dx = \int_{30}^{50} x \cdot \frac{75}{x^2}\,dx =
\]
\[
[75 \log(x)]_{x=30}^{50} = 75 \log(50) - 75 \log(30) \approx 38.1.
\]

Let \( X_1, X_2, X_3 \) denote the three throws, and let \( Y = \max(X_1, X_2, X_3) \).

We aim to compute the cdf of \( Y \):
\[
F_Y(x) = \mathbb{P}(Y < x).
\]
Problem 7

To compute $\mathbb{P}(Y < x)$, first note that

$$\{Y < x\} = \{\max(X_1, X_2, X_3) < x\} = \{X_1 < x, X_2 < x, X_3 < x\}.$$
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But $X_1$, $X_2$ and $X_3$ are independent, so

$$\mathbb{P}(X_1 < x, X_2 < x, X_3 < x) = \mathbb{P}(X_1 < x)\mathbb{P}(X_2 < x)\mathbb{P}(X_3 < x) =$$

$$= \begin{cases} 0 & x \leq 30 \\ \left(\frac{75}{30} - \frac{75}{x}\right)^3 & 30 < x \leq 50 \\ 1 & x > 50. \end{cases}$$