Stochastics
Problem sheet 3 - Generating functions
Fall 2021

1. A nonnegative discrete random variable has generating function

$$G(z) = \frac{3}{8} + \frac{3}{8}z + \frac{1}{8}z^2 + \frac{1}{8}z^3.$$

   Determine the distribution of X (that is, the $\mathbb{P}(X = k)$ probabilities for $k = 0, 1, 2, \dots$). Calculate its mean and variance as well.

2. Alice sends a letter to Bob. Postal service is not very reliable; each day, the postman will take the letter to the logistics center with probability 1/3 (regardless of the past). Once the letter is in the logistics center, each day it is processed with probability 1/5 (regardless of the past). Once it is processed, shipping it takes 1 day. (So at best, the total delivery time is 1 day.) Let X denote the total delivery time in days. Calculate the generating function and the mean of X.

3. An exam has two parts, A and B. Part B of the exam may be taken only by students who pass part A. Each student passes part A with probability 0.6, independent of the others. Each student who passed part A then passes part B with probability 0.5, independent of the others. 100 students take this test. Let X denote the number of students who pass part A, and Y denote the number of students who pass part B. What is the distribution of X? Calculate G_X, the generating function of X, then derive G_Y, the generating function of Y using G_X. Can we tell the distribution of Y from G_Y?

4. We roll a fair six-sided die until we get two consecutive sixes. Let Y denote the number of rolls needed to get two consecutive sixes (so for the sequence 1462655661, $Y = 9$). Calculate the generating function of Y, then the mean of Y. Hint: the theorem of total expectation applies for the generating function.

5. Let $X_1, X_2, \dots$ be iid random variables and N a discrete random variable, independent from the X's, and let $Y = \max(X_1, \dots, X_N)$. Express the cumulative distribution function of Y using the common cumulative distribution function of the X's and the generating function of N.

6. On average, 2 cars per minute pass on a low traffic road. On average, 20% of the cars are red. What is the distribution of the number of red cars passing in a 1 minute interval? (Hint: let X denote the total number of cars passing in the 1 minute interval, and let Y denote the number of red cars passing in the same 1 minute interval. Write Y as a sum with a random number of terms.)

7. * Design two fair 6-sided dice X_1 and X_2 such that $X_1 + X_2$ has the same distribution as the sum of 2 rolls with a regular fair 6-sided die, both are fair, all sides have nonnegative integer numbers, but X_1 and X_2 are different.