2. A certain type of cookies contain on average 3 chocolate chips per cookie and 2 raisins per cookie.

(a) What is the probability that a random cookie will contain exactly 2 chocolate chips?
(b) What is the probability that a random cookie will contain no raisins?
(c) Assuming that a cookie contains a total of 2 pieces (of either chocolate chips or raisins), what is the conditional probability that both of them are chocolate chips?
(d) Joe eats half of a cookie. What is the probability that it contains at least 1 raisin?
(e) Joe eats the second half of the cookie too. What is the conditional probability that the entire cookie contains at least 2 raisins, assuming that the first half contained at least 1 raisin?

Solution.

(a) The number of chocolate chips in a cookie $X$ has distribution POI(3), so $P(X = 2) = \frac{3^2}{2!} e^{-3}$.

(b) The number of raisins in a cookie $Y$ has distribution POI(2), so $P(Y = 0) = \frac{2^0}{0!} e^{-2}$.

(c) Let $X$ be the number of chocolate chips and $Y$ be the number of raisins in a cookie. Then $X \sim$ POI(3), $Y \sim$ POI(2) and $X + Y \sim$ POI(5). The question is

$$P(X = 2 | X + Y = 2) = \frac{P(X = 2 \text{ and } X + Y = 2)}{P(X + Y = 2)} = \frac{\frac{3^2}{2!} e^{-3} \cdot \frac{2^0}{0!} e^{-2}}{\left(\frac{3}{5}\right)^2}.$$

Another solution is that each piece is a chocolate piece with probability $\frac{3}{5}$ (independently from other pieces) since there are 3 chocolate chips on average in a cookie, and 5 pieces in total in average in a cookie. So the probability that both pieces are chocolate chips is $(\frac{3}{5})^2$.

(d) The number of raisins in half a cookie $Z$ has distribution POI($\frac{1}{2} \cdot 2$), so

$$P(Z \geq 1) = 1 - P(Z = 0) = 1 - \frac{1^0}{0!} e^{-1}.$$

(e) $A$ is the event that the entire cookie contains at least 2 raisins. $B$ is the event that the first half of the cookie contains at least 1 raisin. The question is $P(A | B) = \frac{P(A \cap B)}{P(B)}$. $A \cap B$ may occur in one of two ways: either the first half contains 1 raisin and the second half contains at least 1 raisin, or the first half contains at least 2 raisins and the second half contains any number of raisins. Thus

$$P(A | B) = \frac{\frac{1^1}{1!} e^{-1} \left(1 - \frac{1^0}{0!} e^{-1}\right) + \frac{1^2}{2!} e^{-2} \cdot 1}{1 - \frac{2^0}{0!} e^{-2} - \frac{2^1}{1!} e^{-2}}.$$

3. On a road, an average of 2 cars per minute pass by. Jack stands next to the road and starts counting cars.

(a) What is the probability that during a 5 minute interval, no cars pass Jack?
(b) What is the probability that during a 4 minute interval, at most 3 cars pass him by?
(c) What is the probability that during a 2 minute interval, 2 cars pass him by, then during the next 2 minutes, no cars pass him by?
(d) On average, 10% of the cars are red. What is the probability that during a 5 minute interval, no red car passes by?
(e) What is the probability that during a 3 minute interval, exactly 1 red car and exactly 2 non-red cars pass by?

Results.
9. In a shop, customers arrive once every 20 minutes on average. Assuming 4 customers arrive between 10:00 and 11:00, what is the conditional probability that 1 customer arrives between 10:00 and 10:20 (and 3 customers arrive between 10:20 and 11:00)?

Solution. Assuming 4 customers arrive in the given interval, their distribution is i.i.d. uniform over the interval, so each customer will arrive with probability $\frac{1}{3}$ (20 minutes per 60 minutes) between 10:00 and 10:20. The number of customers arriving between 10:00 and 10:20 is then $\text{BIN}(4, \frac{1}{3})$ and the probability in question is $\frac{4 \cdot \left(\frac{1}{3}\right) \cdot \left(\frac{2}{3}\right)^3}{\left(\frac{2}{3}\right)^4} = \frac{32}{81}$.

10. Otto’s car has 2 headlamps (left and right). The left headlamp fails on average once every 12 months, but the right headlamp fails on average once every 4 months due to a persistent error. Headlamps fail independently of each other and the past. When a headlamp fails, it is repaired instantly.

(a) What is the average time between any headlamp failures?
(b) What is the probability that the right headlamp fails next?
(c) Calculate the probability that both headlamps work throughout the entire winter without failure (winter is 3 months).
(d) Calculate the probability that during the winter, all failures occur on the right headlamp.

Solutions. The rate of left headlamp failures is 1 (per year) and right headlamp failures is 3. The union of the two processes means that the rate of headlamp failures is 4, and any failure will be on the right headlamp with probability $\frac{3}{4}$.

(a) $\frac{1}{4}$ years, which is 3 months.
(b) $\frac{3}{4}$.
(c) Let $X$ denote the number of failures over 3 months. Then $X \sim \text{POI}(4 \cdot \frac{1}{4})$, and $P(X = 0) = \frac{e^{-1}}{1!} = e^{-1}$. Alternatively, let $T$ denote the time of the first failure (after winter starts); then $T \sim \text{EXP}(4)$, and the same event is described by $P(T > 1/4) = 1 - F_T(3/4) = 1 - (1 - e^{-4 \cdot \frac{3}{4}}) = e^{-1}$.
(d) This is equivalent to saying that the left headlamp has 0 failures. If $X$ denotes the number of failures on the left headlamp over the winter, then $X \sim \text{POI}(1/4)$ and $P(X = 0) = \frac{(1/4)^0}{0!} e^{-1/4}$. 

\[
\begin{align*}
(a) & \quad \frac{10^6}{6!} e^{-10} \\
(b) & \quad \frac{8^6}{6!} e^{-8} + \frac{8^1}{1!} \frac{1}{2^8} e^{-8} + \frac{8^2}{2!} \frac{1}{2^8} e^{-8} + \frac{8^3}{3!} \frac{1}{2^8} e^{-8} \\
(c) & \quad \frac{4^2}{2!} e^{-4} - \frac{4^0}{0!} e^{-4} \\
(d) & \quad \frac{1^2}{2!} e^{-1} \\
(e) & \quad \frac{(3/5)^1}{1!} e^{-3/5} \cdot \frac{(27/5)^2}{2!} e^{-27/5}
\end{align*}
\]