8. The teacher gives Jack problems to solve. Jack can solve each problem correctly with probability \(1/2\), independently from the others. For each incorrect solution, the teacher gives him two more problems to solve. Jack starts with a single problem to solve. What is the probability that Jack will eventually finish solving the problems? What is the expected number of problems he has to solve before he finishes?

Solution. Define a branching process where individuals are the problems Jack is given, and each problem has 2 offspring if Jack’s solution is wrong and 0 if the solution is correct.

\[ G(z) = \frac{1}{2} + \frac{1}{2}z^2, \] and \( G'(1) = 1 \), so this is a critical process. That means that the probability of extinction is 1, so Jack will eventually finish with probability 1, but \( \mathbb{E}(N) = \infty \), so the expected number of problems he has to solve is infinite.

10. Let \( \Theta(p) \) denote the probability of extinction for a branching process with offspring distribution \( \text{PGEO}(p) \). Calculate the function \( \Theta(p) \) for \( 0 < p < 1 \).

Solution.

\[ G(z) = \sum_{k=0}^{\infty} p(1 - p)^k z^k = \frac{p}{1 - (1 - p)z}. \]

The solutions of \( G(z) = z \) are 1 and \( \frac{p}{1-p} \). For \( p < 1/2 \), the smallest nonnegative root is \( \frac{p}{1-p} \), otherwise it is 1, so

\[ \Theta(p) = \begin{cases} \frac{p}{1-p} & \text{for } 0 < p < 1/2 \\ 1 & \text{for } 1/2 \leq p < 1 \end{cases} \]