Stochastics
Problem sheet 5 - Poisson processes
Fall 2021

1. A call center receives an average of 8 local and 2 long-distance calls during 5 minutes.
    (a) What is the probability that during 2 minutes, they receive exactly 1 long-distance call?
    (b) What is the probability that during 2 minutes, they receive at most 3 calls in total?
    (c) What is the conditional probability that during 2 minutes, they receive exactly 1 long-distance call, assuming that during the same period of time, they receive at most 3 calls in total?
    (d) We start logging calls at $t = 0$. What is the distribution and the mean of the time of the first local call?
    (e) What is the distribution and the mean of the time of the first call (of any type)?
    (f) What is the probability that the next call is local?

2. A certain type of cookie contains on average 3 chocolate chips per cookie and 2 raisins per cookie.
    (a) What is the probability that a random cookie will contain exactly 2 chocolate chips?
    (b) What is the probability that a random cookie will contain no raisins?
    (c) Assuming that a cookie contains a total of 2 pieces (of either chocolate chips or raisins), what is the conditional probability that both of them are chocolate chips?
    (d) Joe eats half of a cookie. What is the probability that it contains at least 1 raisin?
    (e) Joe eats the second half of the cookie too. What is the conditional probability that the entire cookie contains at least 2 raisins, assuming that the first half contained at least 1 raisin?

3. On a road, an average of 2 cars per minute pass by. Jack stands next to the road and starts counting cars.
    (a) What is the probability that during a 5 minute interval, no cars pass Jack?
    (b) What is the probability that during a 4 minute interval, at most 3 cars pass him by?
    (c) What is the probability that during a 2 minute interval, 2 cars pass him by, then during the next 2 minutes, no cars pass him by?
    (d) On average, 10% of the cars are red. What is the probability that during a 5 minute interval, no red car passes by?
    (e) What is the probability that during a 3 minute interval, exactly 1 red car and exactly 2 non-red cars pass by?
    (f) What is the probability that during a 5 minute interval, all cars passing Jack are red?

4. Two types of jobs arrive at a server: type A and type B. On average, the arrival rates are 1 job/second for type A and 2 jobs/second for type B.
    (a) What is the probability that the first job arriving is of type A?
    (b) What is the distribution of the waiting time before the first arrival?
    (c) What is the distribution of the number of type B jobs that arrive before the first type A job?

5. In a shop, customers arrive once every 20 minutes on average. Assuming 4 customers arrive between 10:00 and 11:00, what is the conditional probability that 1 customer arrives between 10:00 and 10:20 (and 3 customers arrive between 10:20 and 11:00)?

6. A 200 page manuscript contains on average 3 typos (errors) per page. During proofreading, 90% of the typos are found and removed.

   (a) The manuscript is 200 pages long. What is the average number of typos remaining in the manuscript *after* proofreading?

   (b) What is the probability that a page originally contained 2 typos, and both are found during proofreading?

   (c) What is the probability that on the first page, all typos are found during proofreading?

7. On average, an office receives a phone call every 5 minutes. For each call, the two operators, Alice and Bob flip a fair coin to decide who takes the call.

   (a) Calculate the probability that exactly two calls arrive between 10:00 and 10:20.

   (b) Calculate the probability that between 10:00 and 10:20, all calls are taken by Alice.

   (c) Assuming that between 10:00 and 10:20, 3 calls arrive in total, what is the conditional probability that Alice has taken exactly two of those calls?

8. We start rolling a fair 6-sided die and add up the results. We keep rolling until the sum reaches (or passes) 1000. What is the distribution of the value of the last roll?

9. In a forest, there are on average 10 trees per $100m^2$. Let us assume that each tree has diameter 20 cm on the ground level. (Ignore the possibility that they may overlap.)

   (a) What is the probability that there are no trees on a given $10m^2$ area?

   (b) We fire a bullet in a random direction from the middle of the forest. What is the probability that the bullet will fly at least 50 meters before it hits a tree?

10. We count trucks on a road. Truck traffic is inhomogeneous, the density of trucks during the day has rate function (number of trucks per hour):

$$\lambda(t) = 6 - 4\cos\left(\frac{\pi}{6}t\right) \qquad t \in [0, 24]$$

   (a) Plot the rate function. At what time are the maximum points?

   (b) What is the average number of trucks passing through the road during one day?

   (c) What is the probability that during 12:00 and 13:00, exactly 3 trucks pass?

HW3 (Deadline: 19 Oct.) Otto's car has 2 headlights (left and right). The left headlight fails on average once every 12 months, but the right headlight fails on average once every 4 months due to a persistent error. Headlights fail independently of each other and the past. When a headlight fails, it is repaired instantly.

   (a) What is the average time between any headlight failures?

   (b) What is the probability that the right headlight fails next?

   (c) Calculate the probability that both headlights work throughout the entire winter without failure (winter is 3 months).

   (d) Calculate the probability that during the winter, all failures occur on the right headlight.