Problem sheet 5
Markov chains
Fall 2019

1. In London, each rainy day is followed by a rainy day with probability 70% and by a sunny day with probability 30%. A sunny day is followed by a rainy day with probability 50% and by a sunny day with probability 50%. Assuming it is raining today, what is the probability that it will be raining 2 days from now? And 3 days from now? And 20 days from now? Calculate the stationary distribution.

2. A drunk man is walking around in a small town. The map of the town is the following:

![Map of the town](map.png)

Whenever the man arrives at any of the corners (A, B, C or D), he will choose his next destination randomly from among the streets available, except the street where he just arrived from.

Is the sequence of corners he visits a Markov chain? If not, propose a Markov chain that describes the situation.

3. Electric Ltd. takes two types of contract jobs: A and B. A type A job lasts for one month and their income from it is 1.4 million HUF, while a type B job lasts for 2 months and their income is 2.7 million HUF. At the beginning of each month, they are open to new contract offers unless they are in the middle of a type B job. At the beginning of each month, they will receive a contract offer for a type B job with 50% probability, while they will receive a contract offer for a type A job with 60% probability (independently from type B offers). If they receive both types of offers, they accept a type A offer.

(a) Model the monthly activity of Electric Ltd. with a Markov-chain. What are the states? What is the transition matrix? Is the Markov chain irreducible? Is it aperiodic?

(b) Calculate the stationary distribution. Based on the stationary distribution, calculate the long-term average monthly income.

(c) What is the average amount of time between consecutive idle months?

(d) They are reconsidering their policy to accept a type A offer when both are available. What is their long-term average monthly income in case they prefer a type B offer when both A and B are available?

4. Janet has 4 scarves: red, brown, orange and yellow. Each day, she selects a scarf at random to wear - except the one she picked the day before. Today she is wearing red.

(a) What is the probability that tomorrow she will wear yellow and the day after that, brown?

(b) What is the probability that 2 days from now, she will wear brown?

(c) Calculate the stationary distribution.

5. A football association has 3 leagues. Pegleg FC starts from league 3. If they are currently in league 3, they get promoted with probability \( \frac{2}{3} \) for the next season. From league 2, they get promoted with probability \( \frac{1}{2} \) for the next season and get relegated with probability \( \frac{1}{6} \) (otherwise, they remain in the current league). From league 1, they get relegated with probability \( \frac{1}{2} \).

(a) Calculate the stationary distribution.

(b) What is the probability that 10 years from now, they will play in league 1?

(c) What is the probability that 10 years from now, they will get relegated at the end of the season?

(d) What is the long term ratio of years they spend in league 2?

(e) Calculate the average number of years that pass between 2 consecutive appearances in league 3.

6. Every morning, Mr. Smith takes home a newspaper with probability \( \frac{1}{3} \) and puts it on the table. In the evening, if there are 3 or more newspapers on the table, Mrs. Smith puts the entire stack of newspapers in the trash. The number of newspapers each day at noon is a Markov process.

(a) What are the states? Calculate the transition matrix.

(b) Calculate the average number of newspapers on the table in the long run.
7. The transition matrix of a Markov chain on state space \{1, 2, 3\} is the following:

\[
\begin{bmatrix}
0 & 1 & 0 \\
2/3 & 0 & 1/3 \\
0 & 1 & 0
\end{bmatrix}
\]

(a) Draw the graph representation of the Markov chain.
(b) Is the Markov chain irreducible? Is it aperiodic?
(c) Calculate the stationary distribution.
(d) Calculate the probability that the Markov chain is in state 1 after 25 steps assuming it is now in state 1.
(e) Calculate the probability that the Markov chain is in state 1 after 25 steps assuming it is now in state 2.
(f) Calculate the long-term average ratio of the steps it spends in state 1.

8. A knight is moving around the squares of the chessboard randomly; the next step is taken uniformly among all possible steps from the current square.

(a) Argue that the position of the knight is a Markov chain.
(b) Is the Markov chain irreducible or not? Is it aperiodic or periodic?
(c) Calculate the stationary distribution.
(d) Assuming the knight is on A1 now, compute the probability that it will be on A1 again after 1000 steps.
(e) Assuming the knight is on A1 now, compute the probability that it will be on A2 after 1000 steps.
(f) Assuming the knight is on A1 now, compute the probability that it will be on A1 again after 1001 steps.
(g) Assuming the knight is on A1 now, compute the probability that it will be on A2 after 1001 steps.

9. John has liability insurance for his car. The insurance company puts drivers into 4 categories: 1, 2, 3, 4. If a driver does not cause any accidents for an entire year, he moves up by 1 category (if he was in category 4, he stays there). If a driver causes a major accident, next year he goes into category 1. If a driver causes a minor accident, but no major accidents during a year, next year he moves down by 1 category (if he was in category 1, he stays there).

John causes a major accident during a year with probability 1/12, and the probability that he causes a minor accident but no major accidents during a year is 1/29.

(a) Model this process with a Markov chain. What are the states? Calculate the transition matrix. Is the Markov chain irreducible? Is it aperiodic?
(b) What is the conditional probability that John will be in category 2 two years from now, assuming that now he is in category 4?
(c) What is the probability that he will be in category 2 ten years from now?
(d) In the long run, how often does he move from category 3 to category 4 on average?
(e) For each category, the annual cost is respectively 120000, 72000, 54000, 36000 HUF. What is the long-term average annual cost paid by John?

HW5 (Deadline: 12 Nov.) Otto is playing a video game which has 3 levels. On level 1, he succeeds with probability 0.8, and proceeds to level 2; otherwise, he has to try level 1 again. On level 2, he succeeds with probability 0.5, and proceeds to level 3; otherwise, he goes to level 1 next. On level 3, he succeeds with probability 0.5. Regardless of the result on level 3, he will go to level 1 next.

(a) Model this process with a Markov chain. What are the states? Calculate the transition matrix. Is the Markov chain irreducible? Is it aperiodic?
(b) What is the conditional probability that 2 games from now, Otto will be playing on level 1, assuming he is playing on level 2 now?
(c) What is the probability that 20 games from now, he will play on level 3?
(d) A game on level 1 takes on average 2 minutes, a game on level 2 takes on average 3 minutes, a game on level 3 takes on average 5 minutes. Calculate the average time of a game.
(e) On average, how many games does he play between two consecutive level 3 successes?