1. In London, each rainy day is followed by a rainy day with probability 70% and by a sunny day with probability 30%. A sunny day is followed by a rainy day with probability 50% and by a sunny day with probability 50%. Assuming it is raining today, what is the probability that it will be raining 2 days from now? And 3 days from now? Calculate the stationary distribution.

2. A corridor is illuminated by 2 light bulbs. At the end of each day, the warden examines the light bulbs. If at least one of the light bulbs is working, he does nothing. If both have failed, he replaces both instantly. Each light bulb fails with probability $\frac{1}{100}$ each day regardless of the past or other light bulbs. Let $X_n$ denote the number of light bulbs working after $n$ days (immediately after inspection and replacement). Is $X_n$ a Markov-chain? What are the states? Calculate the stationary distribution.

3. At an oral exam, the possible marks are $\{1, 2, 3, 4, 5\}$. Each student starts from 3, and the professor starts giving questions. For each correct answer, the student moves up by one, and for each wrong answer, down by one. If the student reaches 1 or 5, the exam ends and the student gets that mark; otherwise, the exam ends after 5 questions, and the student gets the current mark. The current student knows the answer to each question with probability $\frac{2}{3}$.

   (a) Is the Markov chain irreducible? If not, what are the communicating classes? Which class is transient and which is recurrent?
   (b) What is the probability that the student will end up with a 5?

4. Janet has 4 scarves: red, brown, orange and yellow. Each day, she selects a scarf at random to wear - except the one she picked the day before. Today she is wearing red.

   (a) What is the probability that tomorrow she will wear yellow and the day after that, brown?
   (b) What is the probability that 2 days from now, she will wear brown?
   (c) Calculate the stationary distribution.

5. The university football association has 3 leagues. Pegleg FC starts from league 3. If they are currently in league 3, they get promoted with probability $\frac{2}{3}$ for the next season. From league 2, they get promoted with probability $\frac{1}{2}$ for the next season and get relegated with probability $\frac{1}{6}$ (otherwise, they remain in the current league). From league 1, they get relegated with probability $\frac{1}{2}$.

   (a) Calculate the stationary distribution.
   (b) What is the probability that 10 years from now, they will play in league 1?
   (c) What is the probability that 10 years from now, they will get relegated at the end of the season?

6. Every morning, Mr. Smith takes home a newspaper with probability $\frac{1}{3}$ and puts it on the table. In the evening, if there are 3 or more newspapers on the table, Mrs. Smith puts the entire stack of newspapers in the trash. The number of newspapers each day at noon is a Markov process.

   (a) What are the states? Calculate the transition matrix.
   (b) Calculate the average number of newspapers on the table in the long run.

7. Concrete Ltd. takes two types of contract jobs: A and B. A type A job lasts for one month and their income from it is 1.4 million HUF, while a type B job lasts for 2 months and their income is 2.7 million HUF. At the beginning of each month, they are open to new contract offers unless they are in the middle of a type B job.

   At the beginning of each month, they will receive a contract offer for a type B job with 50% probability, while they will receive a contract offer for a type A job with 60% probability. If they receive both types of offers, they accept a type A offer.

   (a) Model the monthly activity of Concrete Ltd. with a Markov-chain. What are the states? What is the transition matrix?
   (b) Calculate the stationary distribution. Based on the stationary distribution, calculate the long-term average monthly income.
   (c) They are reconsidering their policy to accept a type A offer when both are available. What is their long-term average monthly income in case they prefer a type B offer when both A and B are available?
8. The transition matrix of a Markov chain on state space \{1, 2, 3\} is the following:

\[
\begin{bmatrix}
0 & 1 & 0 \\
2/3 & 0 & 1/3 \\
0 & 1 & 0
\end{bmatrix}
\]

(a) Draw the graph representation of the Markov chain.
(b) Is the Markov chain irreducible? Is it aperiodic?
(c) Calculate the stationary distribution.
(d) Calculate the probability that the Markov chain is in state 1 after 25 steps assuming it is now in state 1.
(e) Calculate the probability that the Markov chain is in state 1 after 25 steps assuming it is now in state 2.
(f) Calculate the long-term average ratio of the steps it spends in state 1.

9. An industrial machine is under heavy use and is inspected at the end of each day. As a result of the inspection, it is put into one of the following categories:
- 1 new
- 2 barely used
- 3 very used
- 4 out of order, needs to be replaced

The transition matrix is the following:

\[
\begin{bmatrix}
0 & 7/8 & 1/16 & 1/16 \\
0 & 3/4 & 1/8 & 1/8 \\
0 & 0 & 1/2 & 1/2 \\
1 & 0 & 0 & 0
\end{bmatrix}
\]

(a) Calculate the stationary distribution.
(b) Daily operating cost in each state is 0, 1000, 3000 and 6000 HUF respectively. Calculate the long-term average daily operating cost.
(c) On average, how long is a machine used before it is replaced?

10. A machine is used every day. By the end of the day, an important component of the machine may break with probability 1/10. If it breaks, they replace the component, which takes two days.

(a) Model the state of the machine using a Markov chain. What are the states? Calculate the transition probabilities.
(b) As long as the machine works, it produces a profit of 300 euros per day. Replacing the component costs 420 euros. Calculate the long term average net profit per day.

HW5 Deadline: 14 Nov. John has liability insurance for his car. The insurance company puts drivers into 4 categories: 1, 2, 3, 4. If a driver does not cause any accidents for an entire year, he moves up by 1 category (if he was in category 4, he stays there). If a driver causes a major accident, next year he goes into category 1. If a driver causes a minor accident, but no major accidents during a year, next year he moves down by 1 category (if he was in category 1, he stays there).

John causes a major accident during a year with probability 1/12, and the probability that he causes a minor accident but no major accidents during a year is 1/4.

(a) Model this process with a Markov chain. What are the states? Calculate the transition matrix. Is the Markov chain irreducible? Is it aperiodic?
(b) What is the conditional probability that John will be in category 2 two years from now, assuming that now he is in category 4?
(c) What is the probability that he will be in category 2 ten years from now?
(d) In the long run, how often does he move from category 3 to category 4 on average?
(e) For each category, the annual cost is respectively 120000, 72000, 54000, 36000 HUF. What is the long-term average annual cost paid by John?