3. In a bank, clients are served at 2 windows. In the client area, at most 5 clients may be present at the same time (including the ones being served). When the client area is full, the security guard turns away further clients without service. A client arrives on average every 5 minutes. Serving a client takes 8 minutes on average. When a client is served, the next client in line goes to the window and service starts immediately. If a client arrives when the client area is empty, he will pick a window at random.

(a) Model this process with a CTMC. Calculate the generator.
(b) Calculate the stationary distribution.
(c) What is the probability that at a random time, 3 clients are in the bank?
(d) What is the long term average number of clients?
(e) In the long run, what is the ratio of potential clients that are turned away due to a full client area?
(f) What is the long term average ratio of time when the administrator at the first window is idle?

Solution.

(a) The states are 0, 1, 2, 3, 4 and 5, and the generator is:

$$Q = \begin{pmatrix}
-\frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 & 0 \\
\frac{1}{5} & -\frac{1}{8} & \frac{1}{8} & 0 & 0 & 0 \\
0 & \frac{1}{5} & -\frac{1}{5} & \frac{1}{8} & 0 & 0 \\
0 & 0 & \frac{1}{5} & -\frac{1}{5} & \frac{1}{8} & 0 \\
0 & 0 & 0 & \frac{1}{5} & -\frac{1}{5} & \frac{1}{8} \\
0 & 0 & 0 & 0 & \frac{1}{5} & -\frac{1}{5}
\end{pmatrix}.$$ 

The main remark is that for states 2, 3, 4 and 5, the service rate is doubled since both administrators are working.

(b) This is a birth-death process, so $\frac{1}{5} x_0 = \frac{1}{8} x_1$ and $\frac{1}{5} x_{i-1} = \frac{2}{5} x_i$ for $i = 2, \ldots, 5$, and the solution is $v_{st} = (0.157, 0.251, 0.201, 0.160, 0.128, 0.103)$.

(c) 0.160.

(d) Due to the ergodic theorem, the long term average number of customers is $0 \cdot x_0 + 1 \cdot x_1 + 2 \cdot x_2 + 3 \cdot x_3 + 4 \cdot x_4 + 5 \cdot x_5 = 2.161$.

(e) 0.103.

(f) $x_0 + \frac{x_1}{2} = 0.351$ since in state 1, the administrator at the first window is free half of the time.
HW6 In a car repair shop, the number of cars can change between 0 and 5. Cars to be repaired arrive according to a Poisson process with rate 1/3 (cars per hour), and if there is room, they enter. As long as there is at least 1 car in the shop, the mechanic keeps working. The time it takes to repair a car is exponentially distributed with mean 4. Let $X(t)$ denote the number of cars in the shop at time $t$.

(a) Calculate the generator of $X(t)$.

(b) Calculate the stationary distribution.

(c) Right now, the shop is empty. What is the probability that one month from now, the shop is full?

(d) The owner of the shop makes a profit of 10000 HUF per hour as long as the mechanic is working; however, the profit is −2000 HUF per hour when the mechanic is idle. What is the long term average net profit of the owner?

(e) What is the average number of cars at the shop?

Solution.

(a) The states are 0, 1, 2, 3, 4 and 5, and the generator is:

$$Q = \begin{pmatrix}
\frac{1}{3} & \frac{1}{4} & 0 & 0 & 0 \\
\frac{1}{3} & \frac{1}{4} & \frac{1}{12} & 0 & 0 \\
0 & \frac{1}{4} & \frac{1}{12} & \frac{1}{4} & 0 \\
0 & 0 & \frac{1}{4} & \frac{1}{12} & \frac{1}{4} \\
0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} \\
0 & 0 & 0 & 0 & \frac{1}{4}
\end{pmatrix}.$$

(b) This is a birth-death process, so $\frac{1}{3} x_{i-1} = \frac{1}{4} x_i$ for $i = 1, \ldots, 5$, and the solution is

$v_{\text{st}} = (0.0722, 0.0962, 0.1283, 0.1711, 0.2281, 0.3041)$.

(c) One month is a long time, so this can be approximated by the stationary distribution, and the probability that the shop is full is 0.3041.

(d) By the ergodic theorem, the average net profit per hour is

$$0.0722 \cdot (-2000) + 0.0962 \cdot 10000 + 0.1283 \cdot 10000 + 0.1711 \cdot 10000 + 0.2281 \cdot 10000 + 0.3041 \cdot 10000 = 9134.$$

(Note that as long as there is at least 1 car in the shop, the mechanic is working, so the shop is turning a profit.)

(e) Again, by the ergodic theorem, the average number of cars is

$$0.0722 \cdot 0 + 0.0962 \cdot 1 + 0.1283 \cdot 2 + 0.1711 \cdot 3 + 0.2281 \cdot 4 + 0.3041 \cdot 5 = 3.3.$$