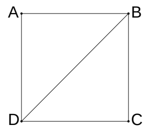
Stochastics
Problem sheet 7 - Markov chains
Fall 2021

1. A drunk man is walking around in a small town. The map of the town is the following:



Whenever the man arrives at any of the corners (A, B, C or D), he will choose his next destination randomly from among the streets available, except the street where he just arrived from.

Is the sequence of corners he visits a Markov chain? If not, propose a Markov chain that describes the situation.

2. Electric Ltd. takes two types of contract jobs: A and B. A type A job lasts for one month and their income from it is 1.4 million HUF, while a type B job lasts for 2 months and their income is 2.7 million HUF. At the beginning of each month, they are open to new contract offers unless they are in the middle of a type B job.

At the beginning of each month, they will receive a contract offer for a type B job with 50% probability, while they will receive a contract offer for a type A job with 60% probability (independently from type B offers). If they receive both types of offers, they accept a type A offer.

   (a) Model the monthly activity of Electric Ltd. with a Markov-chain. What are the states? What is the transition matrix? Is the Markov chain irreducible? Is it aperiodic?
   (b) Calculate the stationary distribution. Based on the stationary distribution, calculate the long-term average monthly income.
   (c) What is the average amount of time between consecutive idle months?
   (d) They are reconsidering their policy to accept a type A offer when both are available. What is their long-term average monthly income in case they prefer a type B offer whenever both A and B are available?

3. Janet has 4 scarves: red, brown, orange and yellow. Each day, she selects a scarf at random to wear - except the one she picked the day before. Today she is wearing red.

   (a) What is the probability that tomorrow she will wear yellow and the day after that, brown?
   (b) What is the probability that 2 days from now, she will wear brown?
   (c) Calculate the stationary distribution.

4. A football association has 3 leagues. Pegleg FC starts from league 3. If they are currently in league 3, they get promoted with probability 2/3 for the next season. From league 2, they get promoted with probability 1/2 for the next season and get relegated with probability 1/6 (otherwise, they remain in the current league). From league 1, they get relegated with probability 1/2.

   (a) Calculate the stationary distribution.
   (b) What is the probability that 10 years from now, they will play in league 1?
   (c) What is the probability that 10 years from now, they will get relegated at the end of the season?
   (d) What is the long term ratio of years they spend in league 2?
   (e) Calculate the average number of years that pass between 2 consecutive appearances in league 3.

5. The transition matrix of a Markov chain on state space $\{1, 2, 3\}$ is the following:

$$\begin{bmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \\ 0 & 1 & 0 \end{bmatrix}$$

   (a) Draw the graph representation of the Markov chain.
   (b) Is the Markov chain irreducible? Is it aperiodic?
   (c) Calculate the stationary distribution.
   (d) Calculate the probability that the Markov chain is in state 1 after 25 steps assuming it is now in state 1.
   (e) Calculate the probability that the Markov chain is in state 1 after 25 steps assuming it is now in state 2.
   (f) Calculate the long-term average ratio of the steps it spends in state 1.

6. A knight is moving around the squares of the chessboard randomly; the next step is taken uniformly among all possible steps from the current square.
   (a) Argue that the position of the knight is a Markov chain.
   (b) Is the Markov chain irreducible or not? Is it aperiodic or periodic?
   (c) Calculate the stationary distribution.
   (d) Compute the conditional probability that the knight will be on A1 after 1000 steps, assuming it is on A1 now.
   (e) Compute the conditional probability that the knight will be on A2 after 1001 steps, assuming it is on A1 now.

7. John has liability insurance for his car. The insurance company puts drivers into 4 categories: 1, 2, 3, 4. If a driver does not cause any accidents for an entire year, he moves up by 1 category (if he was in category 4, he stays there). If a driver causes a major accident, next year he goes into category 1. If a driver causes a minor accident, but no major accidents during a year, next year he moves down by 1 category (if he was in category 1, he stays there).

   John causes a major accident during a year with probability $1/12$, and the probability that he causes a minor accident but no major accidents during a year is $1/4$.
   (a) Model this process with a Markov chain. What are the states? Calculate the transition matrix. Is the Markov chain irreducible? Is it aperiodic?
   (b) What is the conditional probability that John will be in category 2 two years from now, assuming that now he is in category 4?
   (c) What is the probability that he will be in category 2 ten years from now?
   (d) In the long run, how often does he move from category 3 to category 4 on average?
   (e) For each category, the annual cost is respectively 120000, 72000, 54000, 36000 HUF. What is the long-term average annual cost paid by John?

8. A machine is used every day. By the end of the day, an important component of the machine may break with probability $1/10$. If it breaks, they replace the component, which takes *two* days.
   (a) Model the state of the machine using a Markov chain. What are the states? Calculate the transition probabilities.
   (b) As long as the machine works, it produces a profit of 300 euros per day. Replacing the component costs 420 euros. Calculate the long term average net profit per day.

9. Otto is playing a video game which has 3 levels. On level 1, he succeeds with probability 0.8, and proceeds to level 2; otherwise, he has to try level 1 again. On level 2, he succeeds with probability 0.5, and proceeds to level 3; otherwise, he goes to level 1 next. On level 3, he succeeds with probability 0.5. Regardless of the result on level 3, he will go to level 1 next.
   (a) Model this process with a Markov chain. What are the states? Calculate the transition matrix. Is the Markov chain irreducible? Is it aperiodic?
   (b) What is the conditional probability that 2 games from now, Otto will be playing on level 1, assuming he is playing on level 2 now?
   (c) What is the probability that 20 games from now, he will play on level 3?
   (d) A game on level 1 takes on average 2 minutes, a game on level 2 takes on average 3 minutes, a game on level 3 takes on average 5 minutes. Calculate the average time of a game.
   (e) On average, how many games does he play between two consecutive level 3 *successes*?

HW5 (Deadline: 9 Nov.) The company where John works puts employees into 3 categories: good, great and excellent. They update the category of each employee at the end of each month:

- if John is in the good category, the next month he will be good with probability $1/2$ and great with probability $1/2$;
- if John is in the great category, the next month he will be good with probability $1/6$, great with probability $1/2$ and excellent with probability $1/3$;
- if John is in the excellent category, the next month he will be in the excellent category with probability $1/2$ and great with probability $1/2$.

   (a) Model the status of John with a Markov chain. What are the states? Calculate the transition probabilities. Is the Markov chain irreducible? Is it aperiodic?
   (b) Right now, he is in the good category. What is the probability that 2 months from now he will be great?
   (c) Right now, he is in the good category. What is the probability that 1 year from now he will be great?
   (d) What is the long term ratio of months John spends in the great category?
   (e) Employees get a bonus based on their category: employees in the good category get a bonus of 90 euros per month, employees in the great category get a bonus of 150 euros per month, and employees in the excellent category get a bonus of 250 euros per month. Compute the long-term average monthly bonus of John.