Problem sheet 7, some solutions
Markov chains
Fall 2021

3. Janet has 4 scarves: red, brown, orange and yellow. Each day, she selects a scarf at random to wear - except the one she picked the day before. Today she is wearing red.

   (a) What is the probability that tomorrow she will wear yellow and the day after that, brown?

   (b) What is the probability that 2 days from now, she will wear brown?

   (c) Calculate the stationary distribution.

   Solution.

   (a)

   $$\mathbf{P}(\text{yellow 1 day from now and brown 2 day from now}|\text{red today}) = 1/9.$$

   (b) The probability transition matrix is

   $$P = \begin{pmatrix} 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 1/3 & 1/3 \\ 1/3 & 1/3 & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 \end{pmatrix},$$

   the initial vector $v(0) = (1, 0, 0, 0)$ and so $v(2) = v(0)P^2 = (1/3, 2/9, 2/9, 2/9)$, and the probability in question is $2/9$.

   (c) The stationary distribution is

   $$v_{\text{st}} = (3/12, 3/12, 3/12, 3/12) = (1/4, 1/4, 1/4, 1/4).$$

8. A machine is used every day. By the end of the day, an important component of the machine may break with probability $1/10$. If it breaks, they replace the component, which takes *two* days.

   (a) Model the state of the machine using a Markov chain. What are the states? Calculate the transition probabilities.

   (b) As long as the machine works, it produces a profit of 300 euros per day. Replacing the component costs 420 euros. Calculate the long term average net profit per day.

   Solution.

   (a) We need three states: 1 - working, 2 - first day of repair, 3 - second day of repair. The probability transition matrix is

   $$P = \begin{pmatrix} 9/10 & 1/10 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

   (b) The stationary distribution is $(x1\, x2\, x3) = (5/6\ 1/6\ 1/6)$, and according to the ergodic theorem, the long term average net profit per day is

   $$5/6 \times 300 + 1/6 \times (-420) = 180 \text{ (euros)}.$$