Stochastics
Problem sheet 6 - Continuous time Markov chains
Fall 2021

1. A machine works for a random time with distribution Exp(0.1) (in hours) before it fails. When it fails, repair starts immediately, and lasts for a random time with distribution Exp(0.9) (in hours), independently from the length of the working period.
   (a) Argue that this system is a continuous time Markov chain. Determine the generator.
   (b) Calculate the stationary distribution. In the long run, what is the ratio of time spent with repair?
   (c) As long as the machine is operating, it produces an income of 60 dollars per hour. The fee of the repairman is 30 dollars per hour. What is the long term average net profit per hour in the long run?
   (d) Determine the transition matrix of the embedded Markov chain.
2. In a bank, clients are served at 2 windows. In the client area, at most 5 clients may be present at the same time (including the ones being served). When the client area is full, the security guard turns away further clients without service. A client arrives on average every 5 minutes. Serving a client takes 8 minutes on average. When a client is served, the next client in line goes to the window and service starts immediately. If a client arrives when the client area is empty, he will pick a window at random.
   (a) Model this process with a CTMC. Calculate the generator.
   (b) Calculate the stationary distribution.
   (c) What is the probability that at a random time, 3 clients are in the bank?
   (d) What is the long term average number of clients?
   (e) In the long run, what is the ratio of potential clients that are turned away due to a full client area?
   (f) What is the long term average ratio of time when the administrator at the *first* window is idle?
3. In a car repair shop, the number of cars can change between 0 and 5. Cars to be repaired arrive according to a Poisson process with rate $1/3$ (cars per hour), and if there is room, they enter. As long as there is at least 1 car in the shop, the mechanic keeps working. The time it takes to repair a car is exponentially distributed with mean 4. Let $X(t)$ denote the number of cars in the shop at time t.
   (a) Calculate the generator of $X(t)$.
   (b) Calculate the stationary distribution.
   (c) Right now, the shop is empty. What is the probability that one month from now, the shop is full?
   (d) The owner of the shop makes a profit of 10000 HUF per hour as long as the mechanic is working; however, the profit is -2000 HUF per hour when the mechanic is idle. What is the long term average net profit of the owner?
   (e) What is the average number of cars at the shop?
4. Tivadar is a freelance programmer. He takes two types of jobs, both of which have random length. A type A job takes on average 1 month, a type B job takes on average 2 months. When Tivadar finishes with a job, he has to wait on average $2/3$ months before he is offered a type A job, and he has to wait on average 1 month before he is offered a type B job. He takes whichever comes first.
   (a) Model the process with a CTMC. What are the states? Calculate the infinitesimal generator.
   (b) Right now, Tivadar is out of a job. What is the approximate probability that he will be offered a type A job during the next 2 days? What is the probability that he will be offered *any* type of job during the next 2 days? (You may assume a month has 30 days.)
   (c) Calculate the stationary distribution. What is the ratio of time Tivadar spends with jobs of type A in the long run?
   (d) Tivadar has a contract fee of 20000 HUF/day for type A jobs and 50000 HUF/day for type B jobs. Calculate his long term average income per day.
   (e) Calculate the transition matrix of the embedded Markov chain.
   (f) In the long run, from among all jobs he takes, what is the ratio of type A jobs?
   (g) Assume that the distribution of the length of a type B job is NOT exponential, but instead each type B job consists of two IID parts which are both exponential. How can this be modelled by a CTMC? Calculate the generator and the stationary distribution for this new process. Describe the difference between this process and the original.
   (h) He decides not to take type A jobs anymore. Calculate his long term average income per day after that.

5. Let $X(t)$ be a CTMC on state space $\{1, 2\}$ and $Y(t)$ be a CTMC on state space $\{a, b, c\}$ with generators

$$Q_X = \begin{bmatrix} -1/2 & 1/2 \\ 1 & -1 \end{bmatrix} \qquad Q_Y = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 3 & 0 & -3 \end{bmatrix}$$

$X(t)$ and $Y(t)$ are running independently parallel to each other. Let $Z(t) = (X(t), Y(t))$. Argue that $Z(t)$ is a CTMC. Calculate the states and generator matrix of $Z(t)$. Calculate the stationary distribution for $X(t)$ and $Y(t)$ and compare it to the stationary distribution of $Z(t)$.

6. On a telecommunications channel, three independent data streams are present. Each stream has two states: ON and OFF. A stream in ON state has speed 1 Mb/s; in OFF, 0 Mb/s. Each stream changes from ON to OFF with rate λ and from OFF to ON with rate μ independently from everything else. Let X_t denote the total speed of the three data streams at time t. Argue that that the Markov property holds for X_t and calculate the stationary distribution.

7. A burglar commits on average 2 robberies per month when not in prison. During each robbery, he gets caught with probability $1/4$ and is sent to prison. He spends on average 6 months in prison, then goes back to robbing immediately after he is released from prison. The average damage per robbery is 2 million HUF.
   (a) Model the process with a CTMC. What are the states? Calculate the infinitesimal generator.
   (b) Calculate the long-term average ratio of time he spends in prison.
   (c) Calculate the long-term average damage he makes per month.
   (d) Right now, he is free. Estimate the probability that he will still be free 10 days from now (10 days is $1/3$ months).

8. An M/M/k queue has k servers. Jobs arrive according to a Poisson process with rate λ. If there is an idle server when a job arrives, one of the servers starts servicing the job immediately; otherwise, the jobs enters the infinite capacity buffer, and waits for one of the servers to finish service. Jobs are served in the order they arrived (that is, this is a FIFO queue). Assume that the jobs arrive according to a Poisson process with rate λ, and that the service time of each job has distribution EXP(μ).

   For what values of λ, μ and k does a stationary distribution exist? If it exists, calculate it. (Hint: use the balance equations.)

9. A dark corridor is lit by a single light that can be turned on by a button at either end of the corridor. When someone enters the corridor with the light off, they turn on the light. The light turns off automatically after 1 minute; while the light is on, pressing the button has no effect. One person arrives to the corridor on average every 4 minutes. Let X_t denote the state of the light at time t.
   (a) What are the possible states? Is X_t a Markov-chain?
   (b) Calculate the long-term ratio of time when the light is on.
   (c) What is the probability that the light is on when a person arrives?
   (d) It takes 20 seconds for a person to cross the corridor. Assuming that he found the light on when arriving to the corridor, what is the probability that he can cross the entire corridor with the light on?

HW6 (Deadline: 23 Nov.) The football team Pegleg FC has 5 strikers. Each striker *that is playing* becomes injured on average once every 4 months. On average, each injury lasts for 2 months. The team always plays with 3 strikers, unless fewer than 3 are healthy, in which case all healthy strikers play. (So, for example, if all 5 strikers are healthy, 3 of them are playing and the other 2 are not. Only the 3 strikers playing may become injured.)
   (a) Model the number of healthy strikers with a CTMC. What are the states? Calculate the generator.
   (b) Assuming right now all strikers are healthy, what is the approximate probability that an injury will happen in the next 10 days? (You may assume 1 month has 30 days.)
   (c) Calculate the stationary distribution.
   (d) What is the probability that they have to play with less than 2 strikers at a random match?
   (e) What is the average number of injured strikers over a long period of the time?