Stochastics
Problem sheet 8 - Continuous time Markov chains, some solutions
Fall 2021

3. In a car repair shop, the number of cars can change between 0 and 5. Cars to be repaired arrive according to a Poisson process with rate 1/3 (cars per hour), and if there is room, they enter. As long as there is at least 1 car in the shop, the mechanic keeps working. The time it takes to repair a car is exponentially distributed with mean 4. Let $X(t)$ denote the number of cars in the shop at time t.

   (a) Calculate the generator of $X(t)$.
   (b) Calculate the stationary distribution.
   (c) Right now, the shop is empty. What is the probability that one month from now, the shop is full?
   (d) The owner of the shop makes a profit of 10000 HUF per hour as long as the mechanic is working; however, the profit is -2000 HUF per hour when the mechanic is idle. What is the long term average net profit of the owner?
   (e) What is the average number of cars at the shop?

   Solution.

   (a) The states are 0, 1, 2, 3, 4 and 5, and the generator is:

$$Q = \begin{pmatrix} -\frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{4} & -\frac{7}{12} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & -\frac{7}{12} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & -\frac{7}{12} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} & -\frac{7}{12} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{1}{4} & -\frac{1}{4} \end{pmatrix}.$$

   (b) This is a birth-death process, so $\frac{1}{3}x_{i-1} = \frac{1}{4}x_i$ for $i = 1, \ldots, 5$, and the solution is

$$v_{\text{st}} = (0.0722, 0.0962, 0.1283, 0.1711, 0.2281, 0.3041).$$

   (c) One month is a long time, so this can be approximated by the stationary distribution, and the probability that the shop is full is 0.3041.
   (d) By the ergodic theorem, the average net profit per hour is

$$0.0722 \cdot (-2000) + 0.0962 \cdot 10000 + 0.1283 \cdot 10000 + 0.1711 \cdot 10000 + 0.2281 \cdot 10000 + 0.3041 \cdot 10000 = 9134.$$

   (Note that as long as there is ate least 1 car in the shop, the mechanic is working, so the shop is turning a profit.)
   (e) Again, by the ergodic theorem, the average number of cars is

$$0.0722 \cdot 0 + 0.0962 \cdot 1 + 0.1283 \cdot 2 + 0.1711 \cdot 3 + 0.2281 \cdot 4 + 0.3041 \cdot 5 = 3.3.$$

12. Cases arrive to a lawyer according to a Poisson process with average rate 1 case/month. The lawyer takes at most 2 cases at a time; as long as she is working on 2 cases, she declines further cases arriving. A case takes 2 months on average.

   (a) Let X_t denote the number of cases she works at time t. What additional assumption do we need to make to ensure X_t is a continuous time Markov chain? Calculate the generator.
   (b) What is the long-term ratio of time when the lawyer is working (on at least 1 case)?
   (c) The lawyer charges 70000 HUF/day for *each* case she is working on. Calculate her long term average daily income.

   Solution.

   (a) The states are 0, 1, 2 according to the number of cases, and

$$Q = \begin{pmatrix} -1 & 1 & 0 \\ 1/2 & -3/2 & 1 \\ 0 & 1/2 & -1/2 \end{pmatrix}.$$

   (b) The stationary distribution is $(x_0\, x_1\, x_2) = \left(\frac{1}{7}\, \frac{2}{7}\, \frac{4}{7}\right)$, and the ratio of time the lawyer is working is $x_1 + x_2 = \frac{6}{7}$.
   (c) According to the ergodic theorem, her long term average daily income is

$$\frac{1}{7} \times 0 + \frac{2}{7} \times 70000 + \frac{4}{7} \times 140000 = 100000 \text{ (HUF)}.$$