Polyhedron modelling and symmetry groups

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Abstract
The theory of the regular and semi-regular polyhedra is a classical topic of geometry. However, in many cases (e.g. the regular and semi-regular star polyhedra), it seams to be a difficult problem to generate their numerical model by a CAD system. In this paper we shall present a method for constructing the data structure of these polyhedra on the basis of their symmetry group. Also a collection of regular and semi-regular solids will be visualized by computer.

1. Introduction

The theory of regular and semi-regular polyhedra is a classical topic of geometry [1, 2, 3]. The five Platonic and the thirteen Archimedian solids are the most famous representatives of this type of polyhedra, but the four regular star solids and numerous other convex and non-convex semi-regular solids also belong to this family. More generally, in this paper we shall consider those polyhedra whose symmetries form the symmetry group of a Platonic solid (or its orientation preserving subgroup).

The regular and semi-regular solids may be interesting in the education of geometry, of course, but some of them take an important role in architecture and fine arts, too. It is a very instructive (but in many cases very exhausting) problem to plot and model them in descriptive and computational geometry. Stereoscopic figures, which show also the inner structure of these solids, were made by Imre Pál[4], moreover we can find the photos of some gypsum and paper models in [1, 3]. Almost everyone, who works with CAD systems, has already created the computer model of some solids of this family with more or less trouble.

In this paper we shall see a simple method for creating the data structure of a polyhedron on the basis of their symmetries. In this way, actually, our method can be generalized easily, because we could choose other finite symmetry groups, not only the symmetry group of the regular solids.

2. Regular, semi-regular and composite solids

Conventionally, let $\{3\}$, $\{4\}$, $\{5\}$, $\ldots$ denote the regular triangle, square, pentagon, $\ldots$, respectively, moreover let $\{\frac{5}{2}\}$ denote the regular star pentagon the so-called pentagram. A solid is said to be regular if its faces are congruent regular polygons surrounding the vertices alike. Then the vertex figures are also regular polygons. A regular solid is denoted by a pair of numbers in braces: $\{p,q\}$ means that the faces are regular p-gons and the vertex figures are regular q-gons. Transposing the role of the faces and vertices we can get the dual pair $\{q,p\}$ of $\{p,q\}$. There are five convex regular solids (the well-known Platonic solids): the tetrahedron $\{3,3\}$, the octahedron $\{3,4\}$, the cube (hexahedron) $\{4,3\}$, the icosahedron $\{3,5\}$, and the dodecahedron $\{5,3\}$. Moreover, there are four regular star solids: $\{3,\frac{5}{2}\}$ (Figure 1), $\{\frac{5}{2},3\}$ (Figure 2), $\{5,\frac{5}{2}\}$, $\{\frac{5}{2},5\}$; the first one is a star icosahedron, and the others are star dodecahedra. The tetrahedron is self-dual, while the other solids form four dual pairs.

Dispensing with the requirement of the congruence or regularity of faces we can get two possible way to define semi-regular solids. In the first case we obtain the 13 well-known Archimedian solids: $(3,6,6)$, $(3,8,8)$, $(3,10,10)$, $(4,6,6)$, $(4,6,8)$, $(4,6,10)$, $(5,6,6)$, $(3,4,3,4)$ (cuboctahedron), $(3,4,4,4)$, $(3,4,5,4)$, $(3,5,3,5)$ (icosidodecahedron), $(3,3,3,3,4)$, (Figure 3) $(3,3,3,3,5)$. In the parentheses the type of faces around a vertex are enumerated. The Archimedian solids are convex, however there are also non-convex polyhedra in this family, for example $(3,\frac{5}{2},3,\frac{5}{2})$ (Figure 4) and $(5,\frac{5}{2},5,\frac{5}{2})$. In the second case, when the faces are not regular, essentially, we obtain the dual pairs of the previous polyhedra. For example the rhombic dodecahedron and the rhombic triacontahedron are the dual pairs of cuboctahedron and the icosidodecahedron, respectively. The faces of these solids are congruent rhombi.

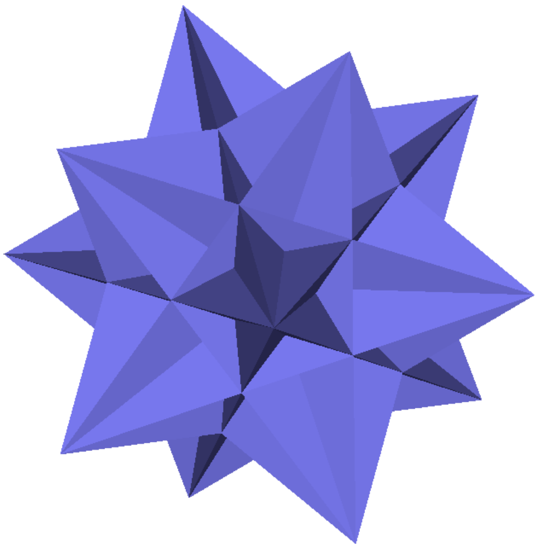

Figure 1: *Regular star icosahedron* $\{3, \frac{5}{2}\}$

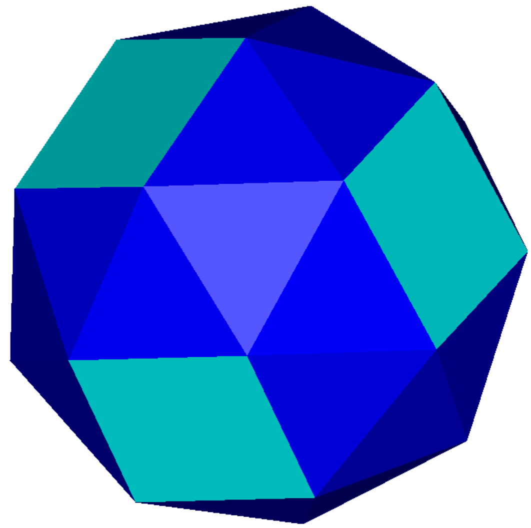

Figure 3: *Archimedian solid* $(3,3,3,3,4)$

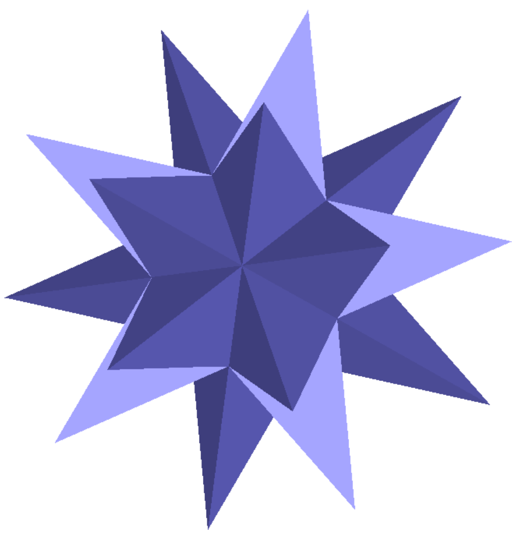

Figure 2: *Regular star dodecahedron* $\{\frac{5}{2}, 3\}$

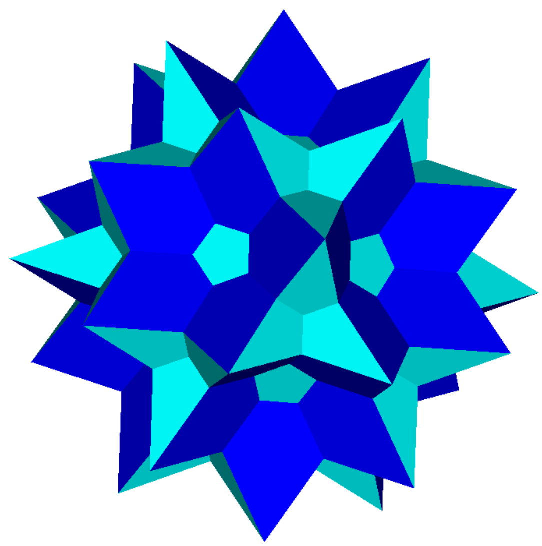

Figure 4: *Semi-regular star solid* $(3, \frac{5}{2}, 3, \frac{5}{2})$

We can obtain newer star polyhedra uniting dual ones. In order to emphasize the main point of duality, we transform the two solids so that their edges intersect each other perpendicularly (Figure 5, 7). In this way, theoretically, we obtain five solids as compositions of the dual pairs (including the pair of tetrahedra), however, in this canonic position $\{\frac{5}{2}, 5\}$ includes its dual pair $\{5, \frac{5}{2}\}$, so that their union is $\{\frac{5}{2}, 5\}$ itself. In similar manner we can construct the union of each Archimedian solid and its dual pair.

In Figure 6 we can see how a cube can be placed in a dodecahedron with common centre and common vertices. Rotating this cube successively 72 degrees about a face normal of the dodecahedron, we can obtain the other four cubes having common vertices with the dodecahedron. The union of these five cubes may be an additional star polyhedron (Figure 11). Replacing the starting cube with its dual octahedron, and rotating this as before, the union of the five octahedra arises. Finally we can start this procedure with one or two tetrahedra, whose edges are the face diagonals of the cube. After rotating we can construct the union of five (Figure 13) or ten tetrahedra, respectively.

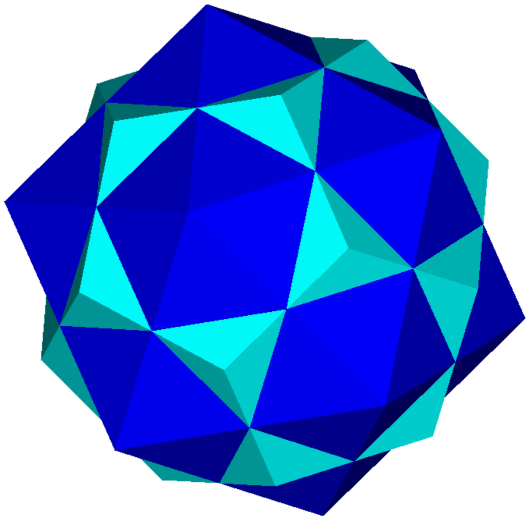

Figure 5: *Dodecahedron and icosahedron*

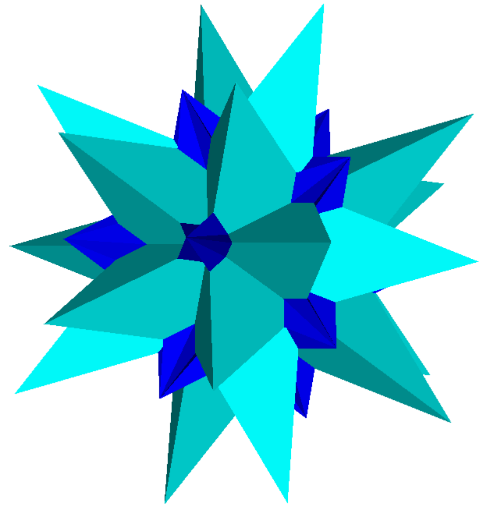

Figure 7: *Star dodecahedron and star icosahedron*

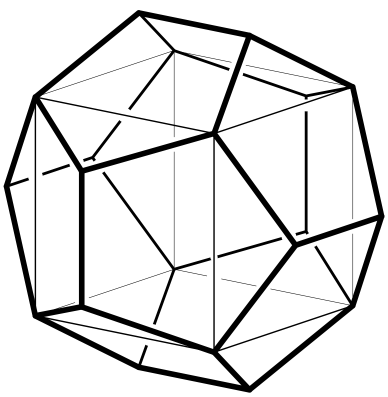

Figure 6: *Cube in dodecahedron*

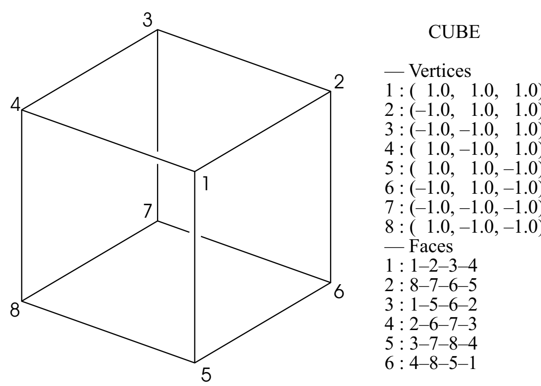

Figure 8: *Data structure of a cube.*

Of course there are several other possibilities for creating polyhedra with many symmetries. Our enumeration contains only some well-known procedures.

3. Data structure and symmetries

In order to describe a solid, first, we enumerate its vertices giving their coordinates. Then we also enumerate the faces as vertex cycles (polygons), in which we refer to the vertices by their index (Figure 8). This simple well-known description can be converted easily to the data structure of other CAD systems.

Basically, we have to consider the groups $\mathcal{T}$, $\mathcal{C}$ and $\mathcal{D}$ containing the 24, 48 and 120 symmetries of the tetrahedron, the cube and the dodecahedron, respectively. Moreover in the case of some polyhedra we shell need the orientation preserving subgroups $\mathcal{T}^+$, $\mathcal{C}^+$ and $\mathcal{D}^+$, too. The symmetry group of each polyhedron enumerated in Section 2 is isomorphic with one of these groups.

In order to obtain these symmetry groups as matrix groups, first, let us create the data structure of the corresponding solids. The vertices of the tetrahedron can be common with the cube: $(1,1,1)$; $(1,-1,-1)$; $(-1,1,-1)$; $(-1,-1,1)$. The vertices of the dodecahedron (Figure 6) are $(\pm1,\pm1,\pm1)$ (these are the vertices of the cube, too); $(0,\pm\tau,\pm\tau^{-1})$; $(\pm\tau^{-1},0,\pm\tau)$; $(\pm\tau,\pm\tau^{-1},0)$, where $\tau = \frac{1}{2}(\sqrt{5}-1)$ the ratio of Golden Section (see also [1, 2]). The vertex cycles of faces can be found easily on the basis of Figure 6. The centre of these polyhedra is the origin of our Cartesian coordinate system. After creating the data structure we compute the midpoint of edges and the centre of faces.

Now, let us consider one of the regular polyhedron P above. We choose a vertex v_0, an edge e_0 and a face f_0 of P,

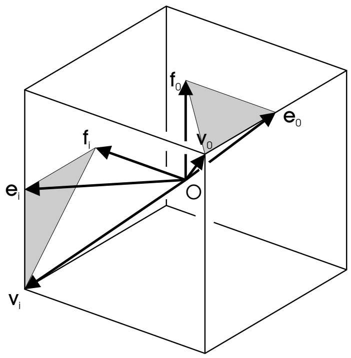

Figure 9: *Searching for the symmetries.*

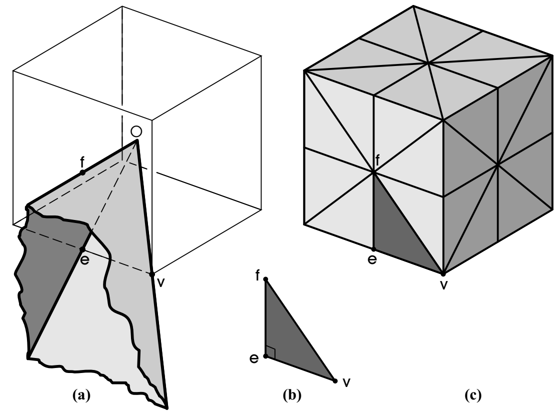

Figure 10: *Fundamental domain of symmetry group $\mathcal{C}$*

which are incident together (in other words e_0 is a side of f_0, and v_0 is an endpoint of e_0). We create the triple $(\mathbf{v}_0, \mathbf{e}_0, \mathbf{f}_0)$ of vectors pointing to v_0, the midpoint of e_0 and the centre of f_0, respectively, as we can see in Figure 9. From the coordinates of these vectors we form the matrix

$$\mathbf{M}_0 = \begin{pmatrix} v_{0x} & e_{0x} & f_{0x} \\ v_{0y} & e_{0y} & f_{0y} \\ v_{0z} & e_{0z} & f_{0x} \end{pmatrix},$$

and compute its inverse $\mathbf{M}_0^{-1}$, too.

Then we enumerate each triple (v_i, e_i, f_i) of vertices, edges and faces of P which are incident together, and create the corresponding vector triple $(\mathbf{v}_i, \mathbf{e}_i, \mathbf{f}_i)$ as above forming the matrix

$$\mathbf{M}_i = \begin{pmatrix} v_{ix} & e_{ix} & f_{ix} \\ v_{iy} & e_{iy} & f_{iy} \\ v_{iz} & e_{iz} & f_{iz} \end{pmatrix}$$

from the coordinates of the vectors. Then the matrix $\mathbf{S}_i = \mathbf{M}_i \mathbf{M}_0^{-1}$ (acting from the left) will describe the transformation that maps the triple $(\mathbf{v}_0, \mathbf{e}_0, \mathbf{f}_0)$ onto $(\mathbf{v}_i, \mathbf{e}_i, \mathbf{f}_i)$. This transformation maps P onto itself, so $\mathbf{S}_i$ is a symmetry of P. At the and of this procedure, we obtain all the symmetries of P. Checking the positive sign of the determinant of matrices we can choose the orientation preserving transformations, which will be necessary to some polyhedra. Finally, we get the symmetry groups $\mathcal{T}$, $\mathcal{C}$ and $\mathcal{D}$, moreover $\mathcal{T}^+$, $\mathcal{C}^+$ and $\mathcal{D}^+$.

The complete symmetry groups $\mathcal{T}$, $\mathcal{C}$ and $\mathcal{D}$ are discrete reflection groups (each can be generated by three plane reflection) fixing the origin [2, 3]. The fundamental domain is a trihedron (an "infinite pyramid") spanned by a vector triple $(\mathbf{v}, \mathbf{e}, \mathbf{f})$ (Figure 10.a). The dihedral angles of the trihedron depend on the group. The images of this trihedral domain under the transformations of the symmetry group tile the whole space without gaps and overlaps.

The intersection of the fundamental trihedron and the surface of the corresponding regular solid is a rectangular triangle (Figure 10.b), whose acute angles, of course, also depend on the group. While the images of the fundamental trihedron tile the space under the symmetry group, the images of this triangle tile the surface of the polyhedron (Figure 10.c).

We remark that the orientation preserving groups $\mathcal{T}^+$, $\mathcal{C}^+$ and $\mathcal{D}^+$ are also discrete transformation groups. Their fundamental domain may be the union of two trihedron having common faces. The edges of the domain are rotation axes, so they must be fixed straight lines, however the faces can be bent because these groups do not contain plane reflections.

4. Constructing the surface of polyhedra on the basis of symmetries

In order to illustrate our method, for example, we create the Archimedean solid $P = (3,8,8)$ (Figure 12.b). First of all, we can recognize that the solid has the same symmetries as the cube. Then we can place it into the cube so that their symmetries will be common metrically, too. We consider the intersection of the fundamental trihedron and the solid surface. So we get a small deltoid (of two right angles) and a rectangular triangle, which adjacent along an edge (Figure 12.a). We form the data structure of an "opened polyhedron" P_0 containing only these two small faces.

The following steps already can be executed algorithmically by a computer program. We apply the transformations of the symmetry group $\mathcal{C}$ for P_0 and enlarge the data structure with its images. Then we obtain a new (closed) polyhedron P_1 whose data structure contains 96 faces which are 48 deltoids and 48 triangles (Figure 12.c). Finally we simplify the structure uniting the adjacent faces in common planes. In our case the result will be the data structure of P describing 6 octagons and 8 triangles.

In this manner, the individual part of our method is to recognize the symmetry group of the polyhedron and compute the common part of its surface and the fundamental trihedron of the group. Usually this problem can also be solved

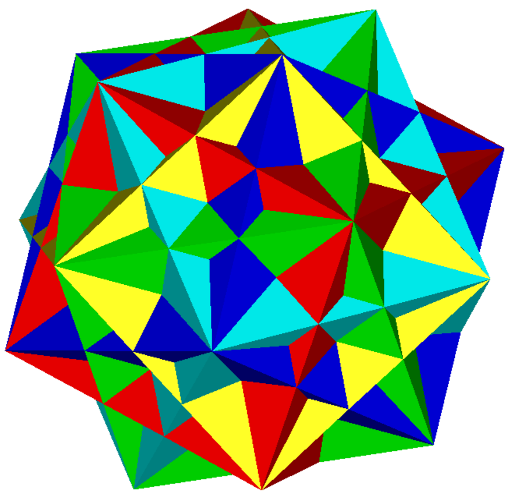

Figure 11: *Five cubes*

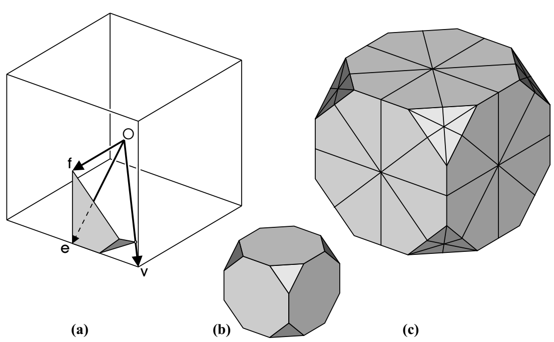

Figure 12: *Creating the Archimedian solid* $(3,8,8)$

easily even for a number of complicated polyhedra. Then the complete surface can be formed by the computer program on the basis of symmetries.

Of course, independently from our algorithm, it is also possible to develop other methods for constructing additional symmetric polyhedra starting with an existing collection of them. For example it seems to be not very difficult to form the dual pair of an Archimedian solid in general. The following step could be to form the union and intersection of dual polyhedra, and so on.

However, our method can be applied not only for creating regular and semi-regular polyhedra. In fact, if we choose a symmetry group and design some adjacent faces freely dividing the fundamental trihedron into a finite and an infinite part, the result of our procedure will be a new symmetric polyhedron. We mention that although $\mathcal{T}$, $\mathcal{C}$ and $\mathcal{D}$ (moreover their orientation preserving subgroups) are the most interesting symmetry groups in our point of view, there are (infinitely) many other discrete transformation groups fixing

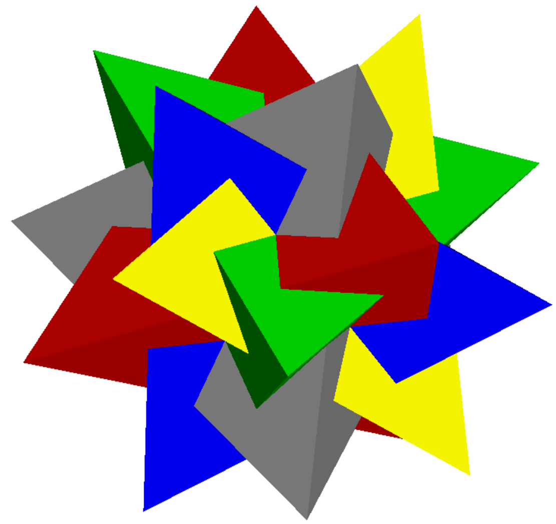

Figure 13: *Five tetrahedra*

the origin. Basically, these are the symmetry groups of regular bipyramids (for more details see [2, 3]). All of them can be applied in a similar manner for creating symmetric polyhedra.

An implementation of this program (as JAVA applet) can be found in the home page of our department: http://www.math.bme.hu/~geom. This program generates the polyhedra in runnung time applying the method above, and the modells can be inspected interactively. The figures of polyhedra in this paper were created also by this program.

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