Bsc Thesis

The Ledrappier - Young formula for diagonal self-affine iterated function systems

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1. Introduction

Ledrappier, Young gave a formula for the Hausdorff dimension of the measures which are invariant for smooth diffeomorphisms on Riemann manifolds. Feng, Hu proved [4] that the Ledrappier-Young formula [9] can be extended to the diagonal self-affine iterated function systems. Since the proof of these theorems are really difficult our object is to represent the most important ideas of the proofs, to give a better understanding about this topic. This will be preceded by a summary of some of the classical results of the dimension theory of self-affine sets and measures.

As a part of the introduction we are going to give a summary about the most common definitions what will be used in the theorems later. We will use the notation of [10].

Definition 1.0.1. *We call the collection of contractive functions* $\{f_i\}_{i=1}^k$ *on* $\mathbb{R}^d$ *an iterated function system, or IFS for short.*

Write $f_{i_1 \dots i_n} := f_{i_1} \dots f_{i_n}$, and let B be the closed ball on $\mathbb{R}^d$. Since for all $i = 1 \dots k$ $\quad f_i$ is contraction, $\{\bigcup_{i_1 \dots i_n} f_{i_1 \dots i_n}(B)\}_{n=1}^{\infty}$ is a nested sequence of sets. Thus we can define the attractor of an IFS in the following way.

Definition 1.0.2. *We define the attractor* Λ *of the IFS* $\{f_i\}_{i=1}^k$ *as*

$$\Lambda := \bigcap_{n=1}^{\infty} \bigcup_{i_1 \dots i_n} f_{i_1 \dots i_n}(B)$$

Since all the theorems are about the Hausforff dimension it is a good idea to define it here, with some other frequently used fractal dimensions.

Definition 1.0.3. *(The Hausdorff dimension) Let* $F \subset \mathbb{R}^d$ *be a non-empty bounded set. We write* $\mathcal{H}^t(F)$ *for the t-dimensional Hausdorff measure of* F*, thus*

$$\mathcal{H}^t(F) := \lim_{\delta \to 0} \left(\inf \left\{ \sum_{i=1}^{\infty} |A_i|^t : F \subset \bigcup_{i=1}^{\infty} A_i, \ |A_i| \leq \delta \right\} \right)$$

Then the Hausdorff dimension of F is the $t = \dim_H(F)$ number where the t-dimensional Hausdorff measure of F drops from infinity to 0.

$$\dim_H(F) := \inf\left\{t : \mathcal{H}^t(F) = 0\right\} = \sup\left\{t : \mathcal{H}^t(F) = \infty\right\}$$

Definition 1.0.4. *(The box dimension) Let $F \subset \mathbb{R}^d$ be a non-empty bounded set, and write $N_\delta(F)$ for the smallest number of δ diameter which can cover F. Then the lower and upper box dimensions of F are*

$$\underline{\dim}_B(F) := \liminf_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$
$$\overline{\dim}_B(F) := \limsup_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

If the limits are equal, then $\dim_B(F) := \underline{\dim}_B(F) = \overline{\dim}_B(F)$.

Definition 1.0.5. *(The similarity dimension) Consider the IFS $\{S_i\}_{i=1}^k$, with contraction ratios $0 < r_i < 1$. Then the similarity dimension of the IFS is the unique $s = \dim_s(F)$ solution of*

$$r_1^s + ... + r_k^s = 1$$

To avoid the consequences of the occurence of heavy overlaps, some theorems may use one of the following assumptions.

Definition 1.0.6. *(The Strong Separation Property (SSP)) The SSP holds for the IFS $\{S_i\}_{i=1}^k$, with attractor Λ, if*

$$S_i(\Lambda) \cap S_j(\Lambda) = \emptyset, \text{ for all } i \neq j$$

Definition 1.0.7. *(The Open SetCondition (OSC)) The OSC holds for the IFS $\{S_i\}_{i=1}^k$, with attractor Λ, if there exists a non-empty open set $V \subset \mathbb{R}^d$ such that*

1. *$S_i(V) \subset V$, for all $i = 1, ..., k$*

2. *$S_i(V) \cap S_j(V) = \emptyset$, for all $i \neq j$*

Other definitions will be given as we proceed.

2. Overlapping constructions

Following [1], in this chapter we show that certain fractals with overlapping cylinders has the same Hausdorff dimension as what we would observe without overlapping, in almost all cases . Then we proof a theorem about the Hausdorff dimension of attractors in some noninjective, picewise linear, "baker's " -type transformation. For both the theorems we will use the same method.

2.1 Hausdorff dimension of overlapping constructions

Definition 2.1.1. *If there exist an E bounded open set, such that*

$$\bigcup_{i=1}^{k} S_i(E) \subset E$$

with the union disjoint, then it means, that the open set condition holds for the $\{S_i\}_{i=1}^k$ iterated function system.

Hutchinson showed that the Hausdorff and the similarity dimensions of a self-similar fractal are equal if the open set condition does hold. Hence

$$\dim E = \dim_s E = s, \quad if \quad \sum_{i=1}^{k} \lambda_i^s = 1 \tag{2.1}$$

where $\dim E$, and $\dim_s E$ denote the Hausdorff and the similarity dimensions of E respectively. In first place we proof that (2.1) gives a good estimate for the Hausdorff dimension, even if the open set condition does not hold, in most cases.

The most important idea of the proof is to convert the fractal into another one in higher dimension for which the open set condition holds. The new fractal must be relatively similar to the original one. Since the open set condition holds for this one, we can calculate the Hausdorff dimension by using (2.1). Then project back the observed invariant set to get an invariant set of the original problem. To make things easier we give the proof in $\mathbb{R}^1$.

Theorem 2.1.1. *(Falconer [1]) Let $T_i : \mathbb{R} \to \mathbb{R}$, $T_i x = \lambda_i x_i$ be linear mappings with contraction ratios $0 < |\lambda_i| < 1$ such that $\sum_{i=1}^{k} |\lambda_i| < 1$. Assume that E is a non-empty compact set satisfying*

$$E = \bigcup_{i=1}^{k} (T_i + c_i)E$$

for almost all $(c_1, ..., c_k) \in \mathbb{R}^k$ in the sense of k-dimensional Lebesgue measure. Then $\dim E = \dim_s E$.

Proof. We define an iterated function system in $\mathbb{R} \times \mathbb{R}^{k-1}$.Let S_i be a linear mapping, with contraction ratio $0 < |\lambda_i| < 1$ defined as follows

$$S_i(x, \mathbf{y}) = (\lambda_i x_i, \lambda_i \mathbf{y} + \mathbf{a}_i).$$

We can see that S_i is the corresponding mapping in $\mathbb{R}^k$ to T_i, it behave like we would have "lifted" T_i to $\mathbb{R}^k$. Thus, S_i are similarity mappings with $|\lambda_i|$ ratios.

Denote the k dimensional hypercube with H, hence $H = (-1, 1)^k$. The condition $\sum_{i=1}^{k} |\lambda_i| = 1$ let us to choose the $\mathbf{a}_i = (a_{i,1}, ..., a_{i,k-1}) \in \mathbb{R}^k$ points in such a way, that the $S_i(H)$ hypercubes are mutually disjoint, and
$S_i(H) \subset H$ holds for all $i = 1, ..., k$. It means, that H satisfies the open set condition for the iterated function system S_i.

Since the open set condition holds, there exists a unique, compact, nonempty F attractor of S_i, hence $F = \cup_{i=1}^{k} S_i(F)$. By 2.1 we also know that $dim F = s$.

Now we need to project this set back to $\mathbb{R}^1$, to get the dimension estimate of the original problem. For that we define an $U_{\mathbf{t}} : \mathbb{R}^k \to \mathbb{R}$ mapping

$$U_{\mathbf{t}}(x_0, \mathbf{x}) = x_0 + \mathbf{x} \cdot \mathbf{t},$$

where $\mathbf{x} \cdot \mathbf{t}$ denotes their scalar product in $\mathbb{R}^(k - 1)$. We can represent the defined mappings between $\mathbb{R}^k$ and $\mathbb{R}$ on the diagram below

$$\begin{array}{ccc} \mathbb{R}^k & \xrightarrow{S_i} & \mathbb{R}^k \\ \downarrow{\scriptstyle U_{\mathbf{t}}} & & \downarrow{\scriptstyle U_{\mathbf{t}}} \\ \mathbb{R}^1 & \xrightarrow{T_i+\mathbf{a}_i\cdot\mathbf{t}} & \mathbb{R}^1 \end{array}$$

It is easy to see that this diagram is commutes $1 \leq i \leq k$, for all $\mathbf{t} \in \mathbb{R}^{k-1}$. By that we get

$$U_{\mathbf{t}}(F) = \bigcup_{i=1}^{k} U_{\mathbf{t}} \circ S_i(F) = \bigcup_{i=1}^{k} (T_i + \mathbf{a}_i \cdot \mathbf{t})(U_t(F))$$

thus $U_{\mathbf{t}}(F)$ is the invariant set for the $\{T_i + \mathbf{a}_i \cdot \mathbf{t}\}_{i=1}^{k}$ system. By the projection theorems for the Hausdorff dimension we get $\dim U_{\mathbf{t}}(F) = \min\{s, 1\}$, for almost all $\mathbf{t} \in \mathbb{R}^{(k-1)}$. We assumed, that $\sum_{i=1}^{k} |\lambda_i| < 1$, it makes the similarity dimension to be lesser than 1, thus $\dim U_{\mathbf{t}}(F) = s$. We observe a similar equation if we translate $U_{\mathbf{t}}(F)$ with an arbitrary $t_0 \in \mathbb{R}$

$$(U_{\mathbf{t}}(F) + t_0) = \bigcup_{i=1}^{k} (T_i + \mathbf{a}_i \cdot \mathbf{t} + (1 - \lambda_i)t_0)(U_{\mathbf{t}}(F) + t_0)$$

The dimension is a translation invariant attribute, so $\dim(U_{\mathbf{t}}(F) + t_0) = s$, for almost all $(t_0, \mathbf{t}) \in \mathbb{R} \times \mathbb{R}^{(k-1)}$. We know, that

$$(t_0, \mathbf{t}) \to (\mathbf{a}_1 \cdot \mathbf{t} + (1 - \lambda_1)t_0, ..., \mathbf{a}_k \cdot \mathbf{t} + (1 - \lambda_k)t_0)$$

is a linear bijection on $\mathbb{R}^k$. Thus, if we use the subtitution

$$\mathbf{a}_i \cdot \mathbf{t} + (1 - \lambda_i)t_0 = c_i,$$

we get that $\dim E = \dim(U_{\mathbf{t}}(F) + t_0) = s$ for almost all $(c_1, ...c_k) \in \mathbb{R}^k$, where E is the unique invariant set for the $\{T_i + c_i\}_{i=1}^{k}$ iterated function system. $\square$

2.2 Extensions of Theorem 2.1.1

We had the strong condition $\sum_{i=1}^{k} |\lambda_i| < 1$ in Theorem 2.1.1. This ensures that $\dim_s E < 1$, where E is the set noted as the attractor in the theorem. It is natural to wonder what happens when that condition does not hold. Simon and Solomyak [11] gave a theorem what covers the unmentioned cases.

Theorem 2.2.1. *(Simon, Solomyak [11]) Let $S_i(x) = \lambda_i x + c_i$, with λ_i indexed in order such as $0 < |\lambda_1| \leq ... \leq |\lambda_k| < 1$. Denote the corresponding attractor with F.*

1. *If $|\lambda_k| + |\lambda_{k-1}| \geq 1$, and S_k, S_{k-1} have different fixed points, then F contains an interval.*
2. $\dim F = \min\{\dim_s F, 1\}$ *for almost all* $(c_1, ..., c_k) \in \mathbb{R}^k$.
3. *If $|\lambda_k| + |\lambda_{k-1}| < 1$, but $\sum_{i=1}^{k} |\lambda_i| > 1$ then $\mathcal{L}(F) > 0$ for almost every $(c_1, ..., c_k) \in \mathbb{R}^k$, where $\mathcal{L}$ denotes the Lebesgue measure.*

The proof of this theorem is based on the proof of theorem 2.1.1, so we only expound it schematically.

Proof. (In steps)

Consider an iterated function system of 2 contractions with their fix-points being distinct. It can be shown that the atractor of such an IFS is a nonempty integral, if $|\lambda_1| + |\lambda_2| > 1$, where λ_i denotes their contraction ratios respectively. From this the proof of 1. follows.

The proof of 2., and 3. goes the same way as the proof of theorem 2.1, we calculate the Hausdorff dimension of a higher dimensional attractor, then project is back to the line. However, this time the similarity dimension > 1, since we assumed $\sum_{i=1}^{k} |\lambda_i| > 1$. According to the projection theorems, the dimension will be $\min\{\dim_s F, 1\}$, and $\dim_s F > 1$, by this this 2. and 3. are also proofed. □

The given dimension estimates only works for almost every $\mathbf{c} = (c_1, ..., c_k) \in \mathbb{R}^k$. The most natural way of generating an exceptional instance with dimension drop, is to make some cylinders coincide. The fat Sierpiński gasket is a good example.

The IFS of the gasket is $\{S_1(x), S_2(x), S_3(x)\} = \{\lambda x + b_1, \lambda x + b_2, \lambda x + b_3\}$, with $\lambda > 1/2$, so all three functions have the same ratio. The translation vectors are $b_1 = (0,0), b_2 = (0,1), b_3 = (1/2, \sqrt{3}/2)$. Choose λ to be the positive solution of the equation $x^3 + x^2 + x = 1$. The coinciding cylinders are

$$S_{1222} \equiv S_{2111}, \quad S_{1333} \equiv S_{3111}, \quad S_{2333} \equiv S_{3222}$$

By giving an upper bound to the Hausdorff dimension with a covering method, we observe

$$\dim F < \dim_s F.$$

2.3 Dimension of attractors of "baker's" -type transformations

The name "baker's" -type comes from a kneading operation what bakers use: they cut the dough in half, then put one of the pieces onto the another one and compress them. These type of transformations are not injective, what may cause trouble about the estimation of the Hausdorff dimension. Since the mapping is not injective, some pieces of the attractor may overlap and that can cause a drop in the Hausdorff dimension. It is a popular belief that the reduction of the dimension is exceptional. We are going to proof that in the case of a certain type of picewise linear transformations the dimension drop is indeed exceptional. In the proof we use a method similar to the one introduced in the proof of theorem 2.1.1.

Now we give the mentioned transformation class. Fix $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{R}$, where $|\lambda_1| + |\lambda_2| < 1$. Let $T_1^{c_1}, T_2^{c_2} : \mathbb{R} \times [-1,1] \to \mathbb{R} \times [-1,1]$ for each $c_1, c_2 \in \mathbb{R}$ be defined by

$$T_1^{c_1}(x,y) = (\lambda_1 x + \mu_1 y + c_1, 2y - 1)$$
$$T_2^{c_2}(x,y) = (\lambda_2 x + \mu_2 y + c_2, 2y + 1)$$

Let $T^(c_1, c_2) : \mathbb{R} \times [-1,1] \to \mathbb{R} \times [-1,1]$ be given by the transformations $T_1^{c_1}, T_2^{c_2}$ in the following way

$$T^{(c_1,c_2)}(x,y) = \begin{cases} T_1^{c_1}(x,y), & \text{if } y \geq 0 \\ T_2^{c_2}(x,y), & \text{if } y < 0 \end{cases}$$

If a compact interval K is large enough, then $T^{(c_1,c_2)}$ will map $K \times [-1,1]$ into itself, and it works for every $(c_1, c_2) \in \mathbb{R} \times \mathbb{R}$. We can obtain the attractor of a transformation of this kind by a decreasing sequence of iterates

$$\bigcap_{i=1}^{\infty} (T^{(c_1,c_2)})^{(i)}(K \times [-1,1]),$$

where $(T^{(c_1,c_2)})^{(i)}$ means that we iterate $T^{(c_1,c_2)}$ i times. This set is independent of the chosen K set.

Theorem 2.3.1. *(Falconer [1]) The invariant set of the transformation $T^{(c_1,c_2)} : \mathbb{R} \times [-1,1] \to \mathbb{R} \times [-1,1]$ defined above has Hausdorff dimension $1 + s$, where $|\lambda_1|^s + |\lambda_2|^s = 1$, and it holds for almost all $(c_1, c_2) \in \mathbb{R} \times \mathbb{R}$.*

Proof. Let $I \subset \mathbb{R}$ be a compact interval such that $\lambda_1 y I + \mu_1 y \subset I$, $\lambda_2 I + \mu_2 y \subset I$ holds for all $-1 \geq y \geq 1$. Now we lift the transformation to a higher dimension just like we did in the proof of theorem 2.1.1. We define the corresponding map S in a similar way to $T^{(c_1,c_2)}$. Let $S_i : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$S_1(x,y,z) = (\lambda_1 x + \mu_1 y, 2y - 1, \lambda_1 z + a_1)$$
$$S_2(x,y,z) = (\lambda_2 x + \mu_2 y, 2y - 1, \lambda_2 z + a_1)$$

The condition $|\lambda_1| + |\lambda_2| < 1$ let us to choose such $a_1, a_2 \in \mathbb{R}$, that $(\lambda_1 I + a_1) \cup (\lambda_2 I + a_2) \subset I$ with the union disjoint, and $a_1(1-\lambda_2) \neq a_2(1-\lambda_1)$. Now we define S with the S_i mappings. Denote it's domain with D, hence $D = I \times [-1,1] \times I \subset \mathbb{R}^3$. Let $S : D \to D$ be

$$S(x,y,z) = \begin{cases} S_1(x,y,z), & \text{if } 0 \geq y \geq 1 \\ S_2(x,y,z), & \text{if } -1 \geq y < 0 \end{cases}$$

We need the set of points in $\mathbb{R}^3$ with the same y coordinate, denote this plane with P_y. We chose I in such a way, that for a given i the iterate $S^{(i)}(D)$ is made of 2^i disjoint paralelepipeds. Thus $P_y \cap S^{(i)}(D)$ consists of 2^i squares with $\lambda_{j_1}, ..., \lambda_{j_i}, \quad j_k = 1, 2$ length sides. Denote the attractor of S with F, hence $F = \cap_{i=1}^{\infty} S^{(i)}(D)$. It can be shown that the Hausdorff dimension of $P_y \cap F = \cap_{i=1}^{\infty}(P_y \cap S^{(i)}(D))$ is s for all $-1 \geq y \geq 1$, where $|\lambda_1|^s + |\lambda_2|^s = 1$.

The next step of our method is to project back the invariant set into $\mathbb{R} \times [-1,1]$. Let $U_t(x,y,z) = (x + tz, y), \quad t \in \mathbb{R}$ be the projection needed. One can see that the following diagram commutes

$$\begin{array}{ccc} D & \xrightarrow{\quad S \quad} & D \\ \big\downarrow{\scriptstyle U_t} & & \big\downarrow{\scriptstyle U_t} \\ \mathbb{R} \times [-1,1] & \xrightarrow{T^{(a_1 t, a_2 t)}} & \mathbb{R} \times [-1,1] \end{array}$$

By that we obtained

$$(T^{(a_1 t, a_2 t)})^{(i)}(U_t(D)) = U_t(S^i)(D)$$

F was defined as the attractor of S, so by taking the limit we observe

$$\bigcap_{i=1}^{\infty} (T^{(a_1 t, a_2 t)})^{(i)}(U_t(D)) = U_t(F)$$

Thus $U_t(F)$ is the attractor of $T^{(a_1 t, a_2 t)}$ for each $t \in \mathbb{R}$.

Both (x,y,z) and $U_t(x,y,z)$ have y as second coordinate, that means $P_y \cap U_t(F) = U_t(P_y \cap F)$. We know that $\dim(P_y \cap F) = s$, from the projection theorems we have

$$s = \dim U_t(P_y \cap F) = \dim(P_y \cap U_t(F))$$

for almost all $t \in \mathbb{R}$.

It can be shown by classic methods that $(y,t) \to \dim U_t(P_y \cap F)$ is a Borel-function. By applying Fubini's theorem for almost all t we observe $\dim(P_y \cap U_t(F)) = s$, for almost all $-1 \leq y \leq 1$. According to Marstand we get the lower bound, $\dim U_t(F) \geq 1 + s$ for almost all $t \in \mathbb{R}$. A covering argument gives us the upper estimate. The set $(T^{(a_1 t, a_2 t)})^{(i)}(D)$ constains $U_t(F)$ for each i, therefore $\dim U_t(F) \leq 1 + s$, since $(T^{(a_1 t, a_2 t)})^{(i)}(D)$ can be covered with 2^i paralellograms of side length $\lambda_{j_1}, ..., \lambda_{j_i}$, $\quad (\lambda_{j_k} = \lambda_1, or \lambda_2)$, where $\sum_{k=1}^{i} |\lambda_{j_k}|^s = 1$. Thus the Hausdorff dimension of the attractor of $T^{(a_1 t, a_2 t)}$ is $1 + s$.

For an arbitrary $t_0 \in \mathbb{R}$

$$(t_0, 0) + T^{(c_1, c_2)}(x - t_0, y) = T^{(c_1 + (1-\lambda_1)t_0, c_2 + (1-\lambda_2)t_0)}(x, y)$$

Because of it, by translating the attractor of $T^{(a_1 t + (1-\lambda_1)t_0, a_2 t + (1-\lambda_2)t_0)}$ we get the attractor of $T^{(a_1 t, a_2 t)}$, so the two has the same dimension. The proof is complete, since it holds for all $(t, t_0) \in \mathbb{R}^2$. $\square$

3. Hausdorff dimension in self-affine cases

In this chapter we will talk about Falconer's theorems about the Hausdorff dimension of self-affine fractals [2]. In the first half we will define the singular value function, then declare the J_∞ index space. After those we can define a measure on the mentioned index set, based on the singular value function, and give bounds on the Hausdorff dimension. At the end of this section we will proof the equality of the Hausdorff and box counting dimensions of self-affine fractals (in a special case). All the theorems and lemmas in this section are from Falconer [2].

3.1 The singular value function

The singular values of a given linear contraction T is the square roots of the eigenvalues of T^*T where T^* is the transpose of T.

Definition 3.1.1. *The singular value function $\phi^s(T)$ is defined by the singular values of the linear map T in the following way*

$$\phi^s(T) = \alpha_1\alpha_2...\alpha_{m-1}\alpha_m^{s-m+1}$$

for each $0 \leq s \leq n$, where m is the smallest integer $s \leq m$.

If $s > n$ we write $\phi^s(T) = (\alpha_1\alpha_2...\alpha_n)^{s/n} = det(T)^{s/n}$.

The singular value function is submultiplicative, continuous, strictly decreasing in s, and if s is an integer with $1 \leq s \leq n$ then

$$\phi^s(T) = \alpha_1...\alpha_s = \sup \mathcal{L}^s(T(E))/\mathcal{L}^s(E),$$

where $\mathcal{L}^s$ is the s-dimensional Lebesgue measure, and the supremum is over all s-dimensional E ellipsoids in $\mathbb{R}^n$. With this function we can estimate the following integral.

Lemma 3.1.1. *Let s be non-integer such that $0 < s < n$. Then there exists a constant $c < \infty$, dependent on n, s and r, such that*

$$I = \int_{B_r} \frac{dx}{|Tx|^s} \leq \frac{c}{\phi^s(T)}$$

for non-singular $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$.

Proof. After some basic transformation on I we have

$$\begin{aligned} I &= \int_{B_r} \frac{dx}{|Tx|^s} = \int_{B_r} \frac{dx}{\langle Tx, Tx\rangle^{s/2}} \\ &= \int_{B_r} \frac{dx}{\langle x, T^*Tx\rangle^{s/2}} \\ &\leq \int \dots \int_{B_r} \frac{dx_1 \dots dx_n}{(\alpha_1^2 x_1^2 + \dots + \alpha_n^2 x_n^2)^{s/2}}, \end{aligned}$$

to get the last estimate we chose coordiante axes in the direction of the eigenvectors of T^*T. With the substitution $y_i = \frac{\alpha_i x_i}{r}$ we have

$$r^{s-n}\alpha_1 \dots \alpha_n I \leq \int \dots \int_P \frac{dy_1 \dots dy_n}{(y_1^2 + \dots + y_n^2)^{s/2}} \tag{3.1}$$

Due to the substitution, the domain of the integrals has changed to P, where $P = \{y = (y_1, \dots, y_n) : |y_i| \leq \alpha_i\}$ is a parallelepiped. Denote the smallest integer bigger then s with m, so $m - 1 < s \leq m$. Since $|y_m| < \alpha_m$ on P, we can use the cover $P \subset P_1 \cup P_2$ to give upper estimate to the integral, define P_1, P_2 as

$$\begin{aligned} P_1 &= \{y \in P : y_1^2 + \dots + y_m^2 \leq 4\alpha_m^2\} \\ P_2 &= \{y \in P : y_1^2 + \dots + y_{m-1}^2 > \alpha_m^2\} \end{aligned}$$

Thus we can continue the estimate (3.1) by dividing the integral into two parts

$$r^{s-n}\alpha_1 \dots \alpha_n I \leq \int \dots \int_{P_1} \frac{dy_1 \dots dy_n}{(y_1^2 + \dots + y_m^2)} + \int \dots \int_{P_2} \frac{dy_1 \dots dy_n}{(y_1^2 + \dots + y_{m-1}^2)}$$

By transforming the coordinates into polar coordinates we get

$$\begin{aligned} r^{s-n}\alpha_1 \dots \alpha_n I &\leq c_1 \alpha_n \dots \alpha_{m+1} \int_0^{2\alpha_m} r^{-s} r^{m-1} dr + c_2 \alpha_n \dots \alpha_m \int_{\alpha_m}^{\infty} r^{-s} r^{m-2} dr \\ &\leq c_3 \alpha_n \dots \alpha_{m+1} \alpha_m^{m-s} + c_4 \alpha_n \dots \alpha_m \alpha_m^{m-s-1} \\ &\leq c_5 \alpha_n \dots \alpha_{m+1} \alpha_m^{m-s}, \end{aligned}$$

where the used constants c_1, c_2, c_3, c_4, c_5 are only dependent on n, s and r. After sorting the inequality we observe the requested estimate for I

$$I \leq \frac{c}{\alpha_1 ... \alpha_{m-1} \alpha_m^{s-m+1}} = \frac{c}{\phi^s(T)}$$

□

3.2 Constructing a measure

We want to define a Hausdorff type measure using the singular value function, so first we need to declare the index set, which will be the range of our measure. Let $J_r = (i_1, ..., i_r) : 1 \leq i_j \leq k$ be the set of sequences of length r, with integers 1 to k, for $r = 1, 2, ...$. Let $J_\infty = (i_1, i_2, ...) : 1 \leq i_j \leq k$ be the set of infinite sequences, and write J for the union of the sets of finite sequences. Deonte the members of J or J_∞ by $\mathbf{i}$, and the length of them by $|\mathbf{i}|$. If $\mathbf{i}, \mathbf{j} \in J_\infty$ then $\mathbf{i} \wedge \mathbf{j}$ is the maximal subsequence contained in both $\mathbf{i}$ and $\mathbf{j}$. We write $N_\mathbf{i}$ for the neighbourhood of $\mathbf{i} \in J_\infty$, thus $N_\mathbf{i} = \{\mathbf{j} \in J_\infty : \mathbf{i} < \mathbf{j}\}$.

If $\mathbf{i} = (i_1, i_2, ...) \in J_\infty$ and $\mathbf{a} = (\alpha_1, ..., \alpha_k) \in \mathbb{R}^{nk}$, write

$$x_\mathbf{i}(\mathbf{a}) = \lim_{r \to \infty} (T_{i_1} + \alpha_{i_1})(T_{i_2} + \alpha_{i_2}) ... (T_{i_r} + \alpha_{i_r})(0) = \alpha_{i_1} + T_{i_1}\alpha_{i_2} + T_{i_1}T_{i_2}\alpha_{i_3} + ...$$

since the T_i are contractions.

Lemma 3.2.1. *If s is non-integer with $0 < s < n$ and $\|T_i\| < \frac{1}{3} \quad (1 \leq i \leq k)$ then there is a $c < \infty$ such that*

$$\int_{\mathbf{a} \in B_r \subset \mathbb{R}^{nk}} \frac{d\mathbf{a}}{|x_\mathbf{i}(\mathbf{a}) - x_\mathbf{j}(\mathbf{a})|^s} \leq \frac{c}{\phi^s(T_{\mathbf{i} \wedge \mathbf{j}})}$$

for all distinct $\mathbf{i}, \mathbf{j} \in J_\infty$

Later Solomyak proofed [12], that this lemma holds for a bigger set of self-affine fractals, with $\|T_i\| < \frac{1}{2} \quad (1 \leq i \leq k)$. According to Edgar the bound 1/2 is sharp.

Proof. (In steps) The main idea of the proof is the integration by substitution. Let $\mathbf{i} \wedge \mathbf{j} = \mathbf{p}$, in this case $\mathbf{i} = \mathbf{p}, \mathbf{i}'$ and $\mathbf{j} = \mathbf{p}, \mathbf{j}'$, for some $\mathbf{i}', \mathbf{j}' \in J_\infty$. First use that $\mathbf{i}$ and $\mathbf{j}$ are share their first $p = |\mathbf{p}|$ coordinates.

$$\int_{\mathbf{a}\in B_r} \frac{d\mathbf{a}}{|x_{\mathbf{i}}(\mathbf{a}) - x_{\mathbf{j}}(\mathbf{a})|^s} = \int_{\mathbf{a}\in B_r} \frac{d\mathbf{a}}{|T_{\mathbf{i}\wedge\mathbf{j}}(x_{\mathbf{i}'}(\mathbf{a}) - x_{\mathbf{j}'}(\mathbf{a}))|^s}$$

By substituting in the integral with $y = x_{\mathbf{i}'}(\mathbf{a}) - x_{\mathbf{j}'}(\mathbf{a})$ and using lemma 3.1.1 we proofed the inequality. □

We define a measure of Hausdorff type on J_∞:

$$\mathcal{M}_r^s(E) = \inf\{\sum_{\mathbf{i}} \phi^s(T_{\mathbf{i}}) : E \subset \bigcup_{\mathbf{i}} N_{\mathbf{i}}, |\mathbf{i}| \geq r\}$$

with an arbitrary positive integer r, and a fix $s \geq 0$.

For each r this is an outer measure, we obtain a net measure of Hausdorff type, by taking the limes in r:

$$\mathcal{M}^s(E) = \lim_{r\to\infty} \mathcal{M}_r^s(E)$$

Now we can define a dimension $d(T_1, ..., T_k)$ with the measure $\mathcal{M}^s$ in an analogous way to the Hausdorff dimension.

Lemma 3.2.2. *The following numbers exist and equal:*
(a) $\inf\{s : \mathcal{M}^s(J_\infty) = 0\} = \sup\{s : \mathcal{M}^s(J_\infty) = \infty\}$
(b) the unique $s > 0$ *satisfying* $\lim_{r\to\infty} \left[\sum_{\mathbf{i}\in J_r} \phi^s(T_{\mathbf{i}})\right]^{1/r} = 1$
(c) $inf\{s : \sum_{\mathbf{i}\in J} \phi^s(T_{\mathbf{i}}) < \infty\} = \sup\{s : \sum_{\mathbf{i}\in J} \phi^s(T_{\mathbf{i}}) = \infty\}$
We denote that value by $d(T_1, ..., T_k)$.

Falconer introduced the affinity dimension as $d(T_1, ..., T_k)$, we can also define this dimension using the topological pressure function.

Definition 3.2.1. *Let* $\mathcal{T} = (T_1, ..., T_k) \in \left(GL_d(\mathbb{R})^k\right)$*, and* $t \geq 0$*, then the topological pressure function is*

$$P(\mathcal{T}, t) := \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{|\mathbf{i}|=n} \phi^t(T_{\mathbf{i}}) \right)$$

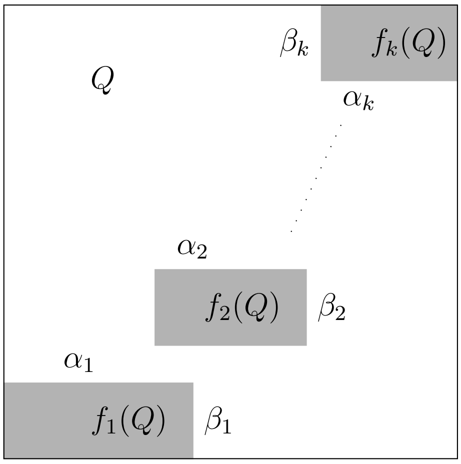

Figure 3.1: The first cylinder of the fractal in the example

For a fix $\mathcal{T}$ this function is strictly decreasing in t, and has an unique zero.

Definition 3.2.2. *The affinity dimension of the self-affine IFS* $\{T_i + t_i\}_{i=1}^k$ *is the unique zero of* $P(\mathcal{T}, t)$.

Example: To show how it works we give the affinity dimension of a separated self-affine fractal in the next example. Consider the IFS $\{f_i = T_i + t_i\}_{i=1}^k$ on $[0,1]^2$, with $T_i = \begin{bmatrix} \alpha_i & 0 \\ 0 & \beta_i \end{bmatrix}$, see figure 3.1. By the definition of the singular value function, for an arbitrary $\mathbf{i}$ we have

$$\phi^s(T_\mathbf{i}) = \begin{cases} \alpha_\mathbf{i}^s, \text{ if } s < 1 \\ \alpha_\mathbf{i}\beta_\mathbf{i}^{s-1}, \text{ if } s \in (1,2) \\ (\alpha_\mathbf{i}\beta_\mathbf{i})^{s/2}, \text{ if } s > 2 \end{cases}, \text{ thus} \sum_{\mathbf{i}\in J_n} \phi^s(T_\mathbf{i}) = \begin{cases} \sum_{\mathbf{i}\in J_n} \alpha_\mathbf{i}^s, \text{ if } s < 1 \\ \sum_{\mathbf{i}\in J_n} \alpha_\mathbf{i}\beta_\mathbf{i}^{s-1}, \text{ if } s \in (1,2) \\ \sum_{\mathbf{i}\in J_n} (\alpha_\mathbf{i}\beta_\mathbf{i})^{s/2}, \text{ if } s > 2 \end{cases}$$

Using these, and the (b) part of Lemma 3.2.2 we can express the affinity dimension with the variables α_i, β_i. We only show the case when $s < 1$, since the other 2 oppurtunities can be seen similarly. If $s < 1$, then $\sum_{\mathbf{i}\in J_n} \phi^s(T_\mathbf{i}) = \sum_{\mathbf{i}\in J_n} \alpha_\mathbf{i}^s$, and from Lemma 3.2.2 (b) we conclude that s must be the unique solution of $\sum_{i=1}^k \alpha_i^s = 1$. In the case of $\sum_{i=1}^k \alpha_i < 1$ we also get this solution, therefore the first case of the formula below follows due to the unicity of s.

$$d(T_1,...,T_k) = \begin{cases} s_0, \text{ where } \sum_{i=1}^k \alpha_i^{s_0} = 1, \text{ if } \sum_{i=1}^k \alpha_i < 1 \\ s_1, \text{ where } \sum_{i=1}^k \alpha_i \beta_i^{s_1-1} = 1, \text{ if } \sum_{i=1}^k \alpha_i \geq 1, \text{ and } \sum_{i=1}^k \alpha_i\beta_i < 1 \\ s_2, \text{ where } \sum_{i=1}^k (\alpha_i\beta_i)^{s_2} = 1, \text{ if } \sum_{i=1}^k \alpha_i\beta_i \geq 1 \end{cases}$$

Note that the Lebesgue-measure of the attractor is 0, if $\sum \alpha_i\beta_i < 1$.

To proof the main theorems about fractal dimensions we will need the following result on net measures, for technical reasons.

Lemma 3.2.3. *Suppose that $\mathcal{M}^s(J_\infty) = \infty$ for an arbitrary s. Then there exist a compact set $E \subset J_\infty$, $0 < \mathcal{M}^s(E) < \infty$ and a constant c such that*

$$\mathcal{M}^s(E \cap N_{\mathbf{i}}) \leq c\phi^s(T_{\mathbf{i}}) \quad (\mathbf{i} \in J)$$

3.3 Theorems about the dimensions

By the following theorems we will show, that the self-affine set $F(\mathbf{a})$ that satisfies $F(\mathbf{a}) = \bigcup_{i=1}^k (T_i + \alpha_i)F(\mathbf{a})$ has Hausdorff dimension equal to this number. First we give an upper bound with the following lemma:

Lemma 3.3.1. *If $\mathcal{M}^s(J_\infty) < \infty$ then $\mathcal{H}^s(F(\mathbf{a})) < \infty$ for all $\mathbf{a} \in \mathbb{R}^{nk}$. In particular $\dim F(\mathbf{a}) \leq d(T_1,...,T_k)$.*

This lemma can be proofed with simple covering argument. Next we give a lower bound.

Lemma 3.3.2. *Suppose that μ is a Borel measure on J_∞ with $0 < \mu(J_\infty) < \infty$ such that for $s < n$*

$$\int_{J_\infty}\int_{J_\infty}\int_{\mathbf{a}\in B_r} \frac{d\mathbf{a}d\mu(\mathbf{i})d\mu(\mathbf{j})}{|x_{\mathbf{i}}(\mathbf{a}) - x_{\mathbf{j}}(\mathbf{a})|^s} < \infty. \tag{3.2}$$

Then for almost all $\mathbf{a} \in B_r \subset \mathbb{R}^{nk}$ (in the sense of nk-dimensional Lebesgue measure) $\dim F(\mathbf{a}) \geq s$.

Proof. We need that $f(\mathbf{a}, \mathbf{i}, \mathbf{j}) = |x_{\mathbf{j}}(\mathbf{a}) - x_{\mathbf{i}}(\mathbf{a})|^{-s}$ is a Borel-measurable function. By taking the limit as $r \to \infty$ of $\min\{r, |x_{\mathbf{j}}(\mathbf{a}) - x_{\mathbf{i}}(\mathbf{a})|^{-s}\}$, which is an increasing sequence of continuous functions, we get $f(\mathbf{a}, \mathbf{i}, \mathbf{j})$, hence f is a non-negative Borel-measurable function. Now we can apply Fubini's theorem on (3.2). By that we get

$$\int_{J_\infty} \int_{J_\infty} \frac{d\mu(\mathbf{i}) d\mu(\mathbf{j})}{|x_{\mathbf{i}}(\mathbf{a}) - x_{\mathbf{j}}(\mathbf{a})|^s} < \infty$$

Then define a measure ν for all $\mathbf{a} \in \mathbb{R}^{nk}$:

$$\nu(E) = \mu\{\mathbf{i} : x_{\mathbf{i}}(\mathbf{a}) \; inE\}.$$

$F(\mathbf{a})$ is the support of this measure, thus $F(\mathbf{a})$ supports a mass distribution of finite s-energy. That means $\dim F(\mathbf{a}) \geq s$, using the potentional-theoretic characterizetion of Hausdorff measure.

□

Now the upper, and lower bounds are given, so we can proof the main theorems about the dimensions.

Theorem 3.3.1. *Assume that $\|T_i\| < \frac{1}{3}$ for $1 \leq i \leq k$. For almost all $\mathbf{a} \in \mathbb{R}^{nk}$ (in the sense of nk-dimensional Lebesgue measure)*

$$\dim F(\mathbf{a}) = \min\{n, d(T_1, ..., T_k)\},$$

where $d(T_1, ..., T_k)$ is the affinity dimension.

The constant was improved by Solomyak [12] in 1998 to $\frac{1}{2}$.

Proof. Let t and s be non-integer numbers such that $0 < t < s <$ min $\{n, d(T_1, ..., T_k)\}$. Then $\mathcal{M}^s(J_\infty) = \infty$, in this case there exest a compact $E \subset J_\infty$ such that $0 < \mathcal{M}^s(E) < \infty$ and $\mathcal{M}^s(E \cap N_{\mathbf{i}}) \leq c\, \phi^s(T_{\mathbf{i}})$ for $\mathbf{i} \in J$. We can define a μ Borel-measure with $\mathcal{M}^s$ by $\mu(A) = \mathcal{M}^s(E \cap A)$. Using this notation we get

$$\mu(N_{\mathbf{i}}) \leq c\, \phi^s(T_{\mathbf{i}})(\mathbf{i} \in J).$$

According to Lemma 3.2.1.

$$\int_{J_\infty}\int_{J_\infty}\int_{B_r}\frac{d\mathbf{a}d\mu(\mathbf{j})d\mu(\mathbf{i})}{|x_\mathbf{i}(\mathbf{a})-x_\mathbf{j}(\mathbf{a})|^s}\le c\int_{J_\infty}\int_{J_\infty}\phi^t\frac{d\mu(\mathbf{i})d\mu(\mathbf{j})}{\phi^t(T_{\mathbf{i}\wedge\mathbf{j}})}.$$

After some further estimations we obtain

$$\int_{J_\infty}\int_{J_\infty}\int_{B_r}\frac{d\mathbf{a}d\mu(\mathbf{j})d\mu(\mathbf{i})}{|x_\mathbf{i}(\mathbf{a})-x_\mathbf{j}(\mathbf{a})|^s}<\infty.$$

That means $\dim F(\mathbf{a})\ge t$ for almost all $\mathbf{a}\in B_r$. We got this with arbitrary r, so $\dim F(\mathbf{a})\ge t$ holds for almost all $\mathbf{a}\in\mathbb{R}^{nk}$. By combining this with the upper bound we obtain the proof of the theorem. □

We obtain the box counting dimension of a given $F\subset\mathbb{R}^n$, by covering it with ϵ sided n-dimensional cubes. The number of cubes we used is $N(\epsilon)$. We will denote the box dimension of F with the limit below

$$\dim_B F=-\lim_{\epsilon\to 0}\log N(\epsilon)/\log\epsilon$$

Theorem 3.3.2. *Assume that* $\|T_i\|<\frac{1}{3}$ *for* $1\le i\le k$. *If* $\dim F(\mathbf{a})=\min\{n,d(T_1,...,T_k)\}$ *then* $\dim_B F(\mathbf{a})$ *exists and equals to* $\dim F(\mathbf{a})$.

Proof. From the definition of Hausdorff dimension we know that $\dim F\le-\underline{\lim}_{\epsilon\to 0}\log N(\epsilon)/\log\epsilon$, for any $F\subset\mathbb{R}$. Let s be an arbitrary $d(T_1,...,T_k)<s<n$ non-integer number, and m be the least $s\le m$ integer. Using the (b) part of Lemma 3.2.2 there may exist an r such that

$$\sum_{\mathbf{i}\in J_r}\phi^s(T_\mathbf{i})\le 1 \tag{3.3}$$

Now we curtain every $\mathbf{i}\in J_\infty$ infinite sequence after rq terms, and denote the curtailment by $\mathbf{i}'$, where q is the smallest integer number for which the mth singular value of $T_{\mathbf{i}'}$ is between the bounds $\epsilon\ge\alpha_m>b^r\epsilon$, with α_m being the mentioned singular value.

Using (3.3) and the subadditivity of the singular value function we get $\sum_{\mathbf{i}':\mathbf{i}\in J_\infty}\phi^s(T_{\mathbf{i}'})\le 1$, since the set of rq-curtailments is a covering of J_∞.

Let B be a ball with diameter $|B| \geq 1$ such that $S_i(B) \subset B$ for every $1 \leq i \leq k$, and $\mathbf{a} \in \mathbb{R}^{nk}$ be a fix parameter. An $S_{\mathbf{i}'}(B)$ ellipsoid can be covered by at most $(4|B|)^n \phi^s(T_{\mathbf{i}'})\alpha_m^{-s}$ cubes of side α_m. It means that to cover $F(\mathbf{a}) \subset \bigcup_{\mathbf{i}'} S_{\mathbf{i}'}(B)$ we may need lesser than

$$(4|B|)^n \sum_{\mathbf{i}'} \phi^s(T_{\mathbf{i}'})\alpha_m^{-s} \leq (4|B|^n)b^{-rs}\epsilon^{-s}$$

cubes of side ϵ. Thus $-\log N(\epsilon)/\log\epsilon \leq (const + s\log\epsilon)/\log\epsilon$.

Overall we observed that

$$-\overline{\lim_{\epsilon\to 0}}\log N(\epsilon)/\log\epsilon \leq s = \dim F \leq -\underline{\lim_{\epsilon\to 0}}\log N(\epsilon)/\log\epsilon,$$

so the proof is complete, since $\dim_B F$ exist. $\square$

4. The box and the affinity dimensions

We will follow [3] in this chapter. By assuming some measure theoretical conditions we will show, that the box dimension, and the affinity dimension of a self-affine fractal are the same. It means, that we have a sufficient condition what makes the Hausdorff dimension to be equal to the affinity dimension.

Since we will need the Lebesgue measure of the projection of a set into a lower dimensional subspace, we want the following condition to hold. For orthogonal projections from $\mathbb{R}^k$ into an arbitrary $(k-1)$ -dimensional subspace Π we use the notation $proj_\Pi$. We denote the k-dimensional Lebesgue-measure with $\mathcal{L}^k$.

Definition 4.0.1. *If the following equality holds for all Π $(k-1)$-dimensional subspaces,*

$$\mathcal{L}^{k-1}(proj_\Pi(U)) = \mathcal{L}^{k-1}(proj_\Pi(\bar{U})) \tag{4.1}$$

then it means U satisfies the projection condition, where U is an open set.

The most technical part in the proof of the equality of the box and affinity dimensions is the use of this measure theoretc lemma. By geometric observations it gives us lower bounds that will be very useful during the proof. To get a better understanding about the notations see figure 4.1.

Lemma 4.0.1. *Let $U \in \mathbb{R}^k$ be a bounded open set, for which the projection condition holds. Assume that there exists an $F \subset \bar{U}$ Borel set and a $c > 0$ constant, such that $proj_\Pi(F) \geq c$ holds for all $(k-1)$-dimensional Π subspaces.*

Then there exists a $\delta > 0$, such that for every Π $(k-1)$- dimensional subspaces there exists a Borel $G \subset proj_\Pi(F)$ satisfying the following conditions:

- $\mathcal{L}^{k-1}(G) \geq \frac{1}{2}c$

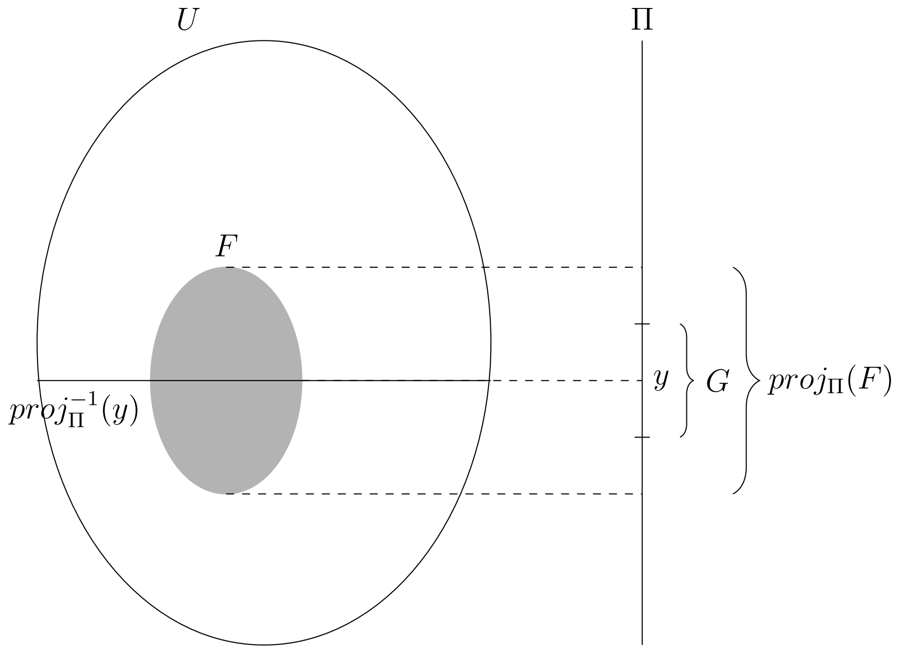

Figure 4.1: Example for Lemma 4.0.1 in 2 dimension

- $\forall y \in G: \quad \mathcal{L}^1\{x \in U : proj_\Pi(x) = y\} \geq \delta$

It is easier to understand this lemma when $k = 2$. If the condition about the Lebesgue-measure of the projections of F does not hold, then $(proj_\Pi(F))$ has zero Lebesgue-measure for some Π 1 dimensional subspace. Thus U can be contained in a line, denote it with l. In this case if we project U to a line perpendicular to l there will be no such G sets, since the intervals $proj^{-1}(y)$ have zero Lebesgue-measure for all possible y.

Proof. Write $B_\delta(x)$ for the x centered closed ball with radius δ, and let $U_\delta = \{x \in U : B_\delta(x) \subset U\}$. Since U is an open set, by using the continuity of the Lebesgue-measure we observe

$$\lim_{\delta \to 0} \mathcal{L}^{k-1}(proj_\Pi(U_\delta)) = \mathcal{L}^{k-1}(proj_\Pi(U)) \tag{4.2}$$

for every Π subspaces. This convergence is uniform in Π as we know from Dini's theorem. Because of 4.2 we can choose such a $\delta > 0$ that

$$\mathcal{L}^{k-1}(proj_\Pi(U_\delta)) \geq \mathcal{L}^{k-1}(proj_\Pi(U)) - \frac{1}{2}c$$

holds for all $(k-1)$- dimensional Π subspaces. For a fix Π let G be defined as

$$G = proj_{\Pi}(U_\delta) \cap proj_{\Pi}(F)$$

This G is a Borel set, and with the principle of inclusion-exclusion we can give the following estimate about it's dimension

$$\mathcal{L}^{k-1}(G) \geq \mathcal{L}^{k-1}(proj_{\Pi}(F)) + \mathcal{L}^{k-1}(proj_{\Pi}(U_\delta)) - \mathcal{L}^{k-1}(\bar{U}) \geq c - \frac{1}{2}c = \frac{1}{2}c$$

It follows from the definition of G that for every $y \in G$ there exists $x \in U_\delta$ such that $proj_{\Pi}(x) = y$. Since $x \in U_\delta$ we have

$$proj_{\Pi}^{-1}(y) \cap B_\delta(x) \subseteq U$$

from this we conclude that $\mathcal{L}^1\{x \in U : proj_{\Pi}(x) = y\} \geq 2\delta \geq \delta$. □

4.1 Sufficient condition for the equality of dimensions

Theorem 4.1.1. *(Falconer [3]) Let F be the attractor of the IFS $\{S_i\}_{i=1}^k$, $S_i : \mathbb{R}^n \to \mathbb{R}^n$. Suppose that both the open set condition and the projection condition hold for a set U, and F satisfies the condition of the previous lemma, thus there exists such a $c \in \mathbb{R}$, $c > 0$ that $\mathcal{L}^{k-1}(proj_{\Pi}(F)) \geq c$ for all $(k-1)$-dimensional Π subspaces. If these hold, the equality of the box and affinity dimensions follows, so*

$$\dim_B F = d(S_1, ..., S_k).$$

In the theorem we used the same notation $d(S_1, ..., S_k)$ for the affinity dimension, as we did in Chapter 3. In the proof we write T_i for the linear part of S_i.

Proof. The conditions in the theorem let us to use lemma 4.0.1, so there exists a $\delta > 0$ that for any Π $(n-1)$-dimensional subspace there exists a

$G \subset proj_\Pi(F)$, $\quad \mathcal{L}^{n-1}(G) \geq \frac{1}{2}c$ for which $\mathcal{L}^1(I_y) \geq \delta$ for all $y \in G$, where I_y denoted the interval $\{x \in U : proj_\Pi(x) = y\}$.

Choose an arbitrary $\mathbf{i} \in J$, where J is the index set defined in Chapter 3. Denote the singular values of $T_\mathbf{i}$ as $\alpha_1(T_\mathbf{i}) \leq \alpha_2(T_\mathbf{i}) \leq ... \leq \alpha_n(T_\mathbf{i})$. By applying $S_\mathbf{i}$ to F each singular value modify the length of F along an axis, these axes are mutually perpendicular. Let the subspace Π to be perpendicular to the axis of the smallest singular value $\alpha_n(T_\mathbf{i})$. The lemma still holds for $T_\mathbf{i}(F)$, we only need to multiply the constants with the corresponding singular values to observe

$$\mathcal{L}^{n-1}(G) \geq \frac{1}{2}c\alpha_1(T_\mathbf{i}), ..., \alpha_{n-1}(T_\mathbf{i}) \tag{4.3}$$

$$\mathcal{L}^1\{x \in S_\mathbf{i}(U) : proj_\Pi(x) = y\} \geq \alpha_n(T_\mathbf{i})\delta \tag{4.4}$$

for all $y \in G$. If $x \in S_\mathbf{i}(U)$ is projected into $proj_\Pi(S_\mathbf{i}(F))$, then x is behind or before F, in the view of Π. Thus, x must be closer to F, then $2R\alpha_n(T_\mathbf{i})$, where R is the radius of the smallest ball in which U fit. By using the maximal distance between F and x, (4.3), and (4.4) we have

$$\begin{aligned} &\mathcal{L}^n\{x \in S_\mathbf{i}(U) : \exists z \in F, |x - z| \leq 2R\alpha_n(T_\mathbf{i})\} \\ &\leq \mathcal{L}^{n-1}(G)\alpha_n(T_\mathbf{i})\delta \leq \frac{1}{2}c\delta\alpha_1(T_\mathbf{i}), ..., \alpha_n(T_\mathbf{i}) \end{aligned}$$

We want a similar estimate about U. To do that we need a Q index set for which the union $\bigcup_{\mathbf{i}\in Q} S_\mathbf{i}(U)$ is disjoint. Let Q be defined as follows

$$Q = \{\mathbf{i} = (i_1, ..., i_j) : \alpha_n(T_\mathbf{i}) \leq r,\ \alpha_n(T_{\mathbf{i}'}) > r,\ \text{where } \mathbf{i}' = (i_1, ...i_{j-1})\}$$

By using that U satisfies the open set condition we observe that the sets $\{S_\mathbf{i}(U)\}_{\mathbf{i}\in Q}$ are disjoint. Similarly as before, we get

$$\begin{aligned}
\mathcal{L}^n\{x \in U : \exists z \in F,\ |x-z| \le 2Rr\} \ge \\
&\ge \sum_{\mathbf{i}\in Q} \mathcal{L}^n\{x \in S_{\mathbf{i}}(U) : \exists z \in F,\ |x-z| \le 2R\alpha_n(T_{\mathbf{i}})\} \\
&\ge \frac{1}{2}c\delta \sum_{\mathbf{i}\in Q} \alpha_1(T_{\mathbf{i}})...\alpha_n(T_{\mathbf{i}}) \\
&\ge \frac{1}{2}c\delta b^{n-s} \sum_{\mathbf{i}\in Q} \alpha_1(T_{\mathbf{i}})...\alpha_{n-1}(T_{\mathbf{i}})\alpha_n(T_{\mathbf{i}})^{1-n+s} r^{n-s}
\end{aligned}$$

for $s \ge 0$, in the end we used that $\alpha_n(T_{\mathbf{i}}) \ge br$ where $b = \min_{1\le i\le k} \alpha_n(T_i)$. We have $\phi^s(T_{\mathbf{i}})$ in the last sum (ϕ^s is the singular value function defined in Chapter 2), and we know that $\sum_{\mathbf{i}\in Q} \phi^s(T_{\mathbf{i}}) \ge m$ if $s < d(S_1, ..., S_k)$, with $m > 0$ independent of r. Hence, in the end of the inequality we have r^{n-s} multiplyed with a constant c', in short

$$\mathcal{L}^n\{x \in U : \exists z \in F, |x-z| \ge 2Rr\} \ge c' r^{n-s}.$$

By rearranging this inequality we observe $s \le \underline{\dim}_B F$. We already showed in the previous chapter, that $\overline{\dim}_B F < d(S_1, ..., S_k)$. Overall we get

$$d(S_1, ..., S_k) \le \underline{\dim}_B F \le \overline{\dim}_B F \le d(S_1, ..., S_k)$$

by that the equality follows. □

Corollary 4.1.1. *Consider the IFS $\{S_i\}_{i=1}^k$ on $\mathbb{R}^2$ with attractor F. If the open set condition holds for a connected set U, and F has a 2 dimensional connected component, then $\dim_B F = d(S_1, ..., S_k)$.*

This corollary is an immediate consequance of the theorem. We only need to realise that the conditions in the corollary make F and U to satisfy the conditions in the theorem.

Proof. Simply, U is a connected subset, so it satisfies the projection condition. F must have some non-collinear points what can form a triangle. One can see that it makes F to satisfy the condition in the theorem about it's $(k-1)$-dimensional projections. □

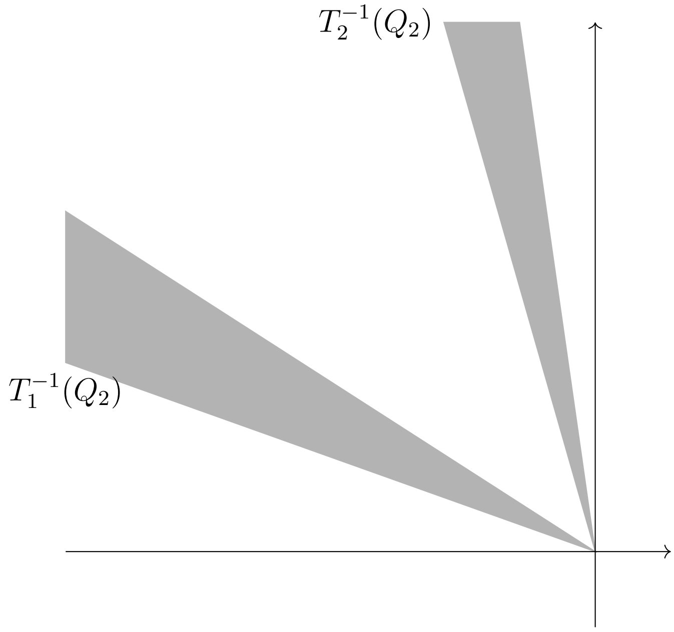

Figure 4.2: The consequence of condition 3.

4.2 A special case of the formula in $\mathbb{R}^2$

Hueter and Lalley [6] also gave sufficient conditions for the equality of the box and affinity dimensions. Consider the IFS $\{S_i = T_i + a_i\}_{i=1}^k$ with attractor F, where T_i is a linear transformation on $\mathbb{R}^2$, and $a_i \in \mathbb{R}^2$ for all $1 \leq i \leq k$.

We want the following conditions to hold:

1. $\forall i: \ ||T_i|| < 1$, where $||\,.\,||$ denotes the matrix norm.
   This ensures that the transformations are contractions.

2. $\forall i: \ \alpha_1(T_i)^2 \leq \alpha_2(T_i)$, where $\alpha_1(T_i) \geq \alpha_2(T_i)$ are the singular values of T_i.

3. Write Q_2 for the closed second quadrant of $\mathbb{R}^2$, then $T_i^{-1}(Q_2) \subset Q_2$, and these subsets are pairwise disjoint.

4. The closed set condition holds for $\{S_i\}_{i=1}^k$.

Theorem 4.2.1. *(Hueter, Lalley [6]) If the conditions above hold then*

$$\dim_H(F) = \dim_B(F) = d(T_1, ...T_k)$$

Furthermore, the d-dimensional Hausdorff-measure of the attractor F is finite, thus

$$\mathcal{H}_d(F) < \infty$$

5. Perturbated attractors

Consider a self-affine iterated function system on $\mathbb{R}^d$. The occurence of some random noise in a system of that kind can be represented if we make small random translations at each use of the transformations. We will give estimates for the Hausdorff dimension in such cases. Throughout this chapter we are going to write $[\omega]$ for the corresponding cylinder in the symbolic space Σ, thus

$$[(i_0, i_1, ..., i_n)] = \{\mathbf{j} \in \Sigma : j_0 = i_0, j_1 = i_1, ..., j_n = i_n\}$$

All the results represented here are from Jordan, Pollicott and Simon [7].

5.1 Notations

Consider a self-affine IFS of the form

$$\{f_i(\mathbf{x}) = T_i(\mathbf{x}) + \mathbf{c}_i\}_{i=1}^{k},$$

where $A_i : \mathbb{R}^d \to \mathbb{R}^d$ is a non-singular contraction, and $\mathbf{c}_i \in \mathbb{R}^d$ a translation vector for all i. Write F for it's invariant set.

Let $\mathbf{y}$ be the vector of the random errors. Suppose that the errors have an absolutely continuous distribution μ, which is supported on an arbitrary small, origin centered set D. Write $F^{\mathbf{y}}$ for the attractor of the perturbated set. The n^{th} cylinder of the original attractor F is defined by all possible n length iteration of the f_i functions. Similarly, to get $F^{\mathbf{y}}$ we need to iterate the perturbated functions. An n^{th} length iteration has the form

$$f_{\mathbf{i}_n}^{\mathbf{y}_{\mathbf{i}_n}} := (f_{i_0} + y_{i_0}) \circ (f_{i_1} + y_{i_0 i_1}) \circ ... \circ (f_{i_{n-1}} + y_{i_0 i_1 ... i_{n-1}}),$$

if we iterate along an $\mathbf{i}_n = (i_0, i_1, ..., i_{n-1}) \in \{1, 2, ..., k\}^n$ index sequence.

We assume that the small translations y_i in the vector $\mathbf{y}$ are independent random variebles with distribution μ. Since D is the support of μ

$$\mathbf{y}_{\mathbf{i}_n} = (y_{i_0}, y_{i_0 i_1}, ..., y_{i_0 i_1 ... i_{n-1}}) \in D \times D \times ... \times D = D^n$$

Let D^∞ be the set of all random errors. We assumed that we know the distribution of the errors, therefore it is practical to use this μ if we want to give a statment about almost all $\mathbf{y} \in D^\infty$. Let $\mathbb{P}$ be the infinite product measure

$$\mathbb{P} := \mu \times \mu \times ... \times \mu \times ...$$

From the definition of the invariant set we observe

$$F^{\mathbf{y}} = \bigcap_{n=0}^{\infty} \bigcup_{\mathbf{i}_n} f_{\mathbf{i}_n}^{\mathbf{y}_{\mathbf{i}_n}},$$

where $B \in \mathbb{R}^d$ is sufficiently large, origin centered ball.

Write $\Pi^{\mathbf{y}} : \Sigma \to \mathbb{R}^d$ for the natural projection from the symbolic space $\Sigma = \{1, 2, ..., k\}^{\mathbb{N}^+}$ to the attractor. Since it is the natural projection we know that

$$\Pi^{\mathbf{y}}(\mathbf{i}) := \lim_{n\to\infty} f_{\mathbf{i}_n}^{\mathbf{y}_{\mathbf{i}_n}}(0),$$

where $\mathbf{i}_n$ is the n length abbrevation of $\mathbf{i} \in \Sigma$.

For a given ergodic measure ν, write h_ν for it's entropy, and
$\lambda_1(\nu) \geq \lambda_2(\nu) \geq ... \geq \lambda_d(\nu)$ for it's Lyapunov exponents.

Definition 5.1.1. *We define the Lyapunov dimension of an ergodic measure in the following way. Let*

$$k = \max\{i : \ 0 < h_\nu + \lambda_1(\nu) + ... + \lambda_i(\nu)\}$$

1. *If $k < d$, then*

$$D(\nu) := k + \frac{h_\nu + \lambda_1(\nu) + ... + \lambda_k(\nu)}{-\lambda_{k+1}(\nu)}$$

2. *If $k = d$, thus $h_\nu + \lambda_1(\nu) + ... + \lambda_d(\nu) > 0$, then*

$$D(\nu) := d \cdot \frac{h_\nu}{-(\lambda_1(\nu) + ... + \lambda_d(\nu))}$$

Figure 5.1 shows what the Lyapunov dimension means. Note that this definition is the extension of the similarity dimension for self-affine cases.

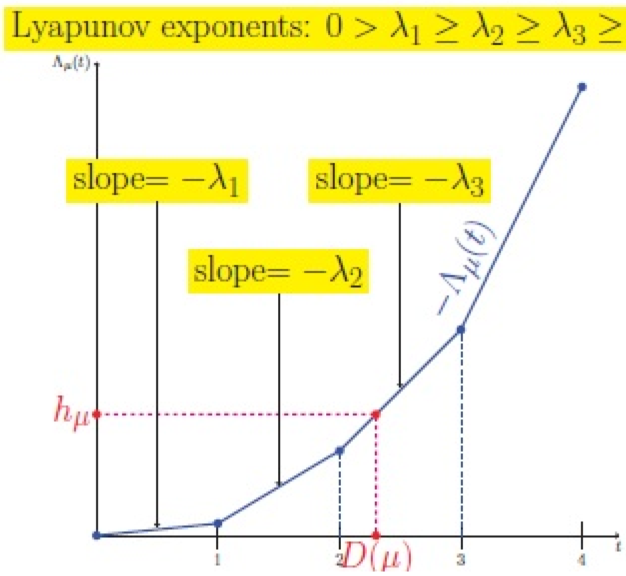

Figure 5.1: This figure from [10] shows us what the definition of the Lyapunov dimension means

5.2 Termodinamical formalism

Definition 5.2.1. *We call the following $P(s)$ function pressure function:*

$$P(s) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{|\mathbf{i}|=n} \phi^s(T_{\mathbf{i}})$$

This is a strictly decreasing function, and has a unique zero. Recently D. J. Feng and P. Shmerkin [5] have proofed that $s \longmapsto P(s)$ is continuous (which was a difficult result).

Definition 5.2.2. *For an arbitrary ergodic measure ν we call the following function $E_\nu(s)$ the energy of ν :*

$$E_\nu(s) := \lim_{n\to\infty} \frac{1}{n} \int \log \phi^s(T_{\mathbf{i}_n}) d\nu(\mathbf{i})$$

where $T_{\mathbf{i}_n} = T_{i_0} ... T_{i_{n-1}}$ if $i_0, ..., i_{n-1}$ are the first n entry of $\mathbf{i} \in \Sigma$.

Lemma 5.2.1. *Consider the IFS* $\{T_i + c_i\}_{i=1}^k$*, and an ergodic measure* ν *on* Σ*. We can write*

$$E_\nu(s) = \begin{cases} \lambda_1(\nu) + ... + \lambda_k(\nu) + (s-k)\lambda_{k+1}(\nu), & if\ s < d \\ \frac{s}{d}(\lambda_1(\nu) + ... + \lambda_d(\nu)), & if s \geq d \end{cases}$$

Using this lemma we observe

$$h_\nu + E_\nu(D(\nu)) = 0$$

The following theorem from Käenmäki shows us the importance of these functions.

Theorem 5.2.1. *For a given* ν *ergodic measure*

$$P(s) \geq h_\nu + E_\nu(s)$$

Furthermore, there exists a μ *ergodic measure on* Σ *such that*

$$0 = P(s_0) = h_\mu + E_\mu(s_0)$$

holds for $s_0 = d(T_1, ..., T_k)$*.*

It follows from this theorem, that the affinity dimension is the root of the pressure function. Thus we can calculate the affinity dimension using $P(s)$, since it's zero is unique.

5.3 Hausdorff dimension with random errors

Now we can state our main theorems by using the notations above. Note that these results do not need any assumptions about the matrix norms.

Theorem 5.3.1. *Let* F *be the attractor of the IFS* $\{T_i + c_i\}_{i=1}^k$*. We have that*

1. *If* $d(T_1, ..., T_k) \leq d$*, then* $\dim_H F^{\mathbf{y}} = d(T_1, ..., T_k)$
2. $d(T_1, ..., T_k) > d$*, then* $\mathcal{L}^d(F^{\mathbf{y}}) > 0$

holds for almost every $\mathbf{y} \in D^\infty$*, in the sense of* $\mathbb{P}$*.*

We show some properties of the image of an ergodic measure in the next theorem. Write $\Pi_*^{\mathbf{y}}$ for the corresponding push-down measure, thus

$$\Pi_*^{\mathbf{y}}(\nu) = \nu((\Pi^{\mathbf{y}})^{-1})$$

for some $\nu : \Sigma \to \mathbb{R}$ measure on the symbolic space Σ.

Theorem 5.3.2. *Let* F *be the attractor of the IFS* $\{T_i + c_i\}_{i=1}^k$*, and* ν *an ergodic measure on* Σ*. We have, that*

1. $\dim_H \Pi_*^{\mathbf{y}}(\nu) = \min\{d, D(\nu)\}$
2. *If* $D(\nu) > d$*, then* $\Pi_*^{\mathbf{y}}(\nu) \ll \mathcal{L}^d$*, where* $\mathcal{L}^d$ *denotes the* d *- dimensional Lebesgue-measure.*

holds for almost all $\mathbf{y} \in D^\infty$ *in the sense of* $\mathbb{P}$

Let U be a compact metric space, $\mathcal{M}$ be a finite Borel measure on U, and $\Omega \subset \Sigma = \{1, ..., k\}^{\mathbb{N}^+}$ be a compact and σ - invariant subset of the symbolic space Σ. Let $\Pi : U \times \Omega \to \mathbb{R}^d$ be the natural projection, where we choose the random error from U. Write $\alpha_k(\mathbf{i}_n)$ for the k^{th} largest singular value of $A_{\mathbf{i}_n} = A_{i_0} \cdot ... \cdot A_{i_n}$, where $\mathbf{i}_n = (i_0, i_1, ..., i_n)$. We will also use this notation in the theorems below.

Definition 5.3.1. *(Self-affine Hölder Condition) For every* $u \in U$, $\mathbf{i} \in \Omega$, $n \in \mathbb{N}$ *there exists a* $K > 0$ *constant, and a* $G(\mathbf{i}, n) : \mathbb{R}^d \to \mathbb{R}^d$ *isometry, that*

$$\Pi^u(\mathbf{i}_n) \subset K \cdot G([0, \alpha_1(\mathbf{i}_n)] \times [0, \alpha_2(\mathbf{i}_n)] \times ... \times [0, \alpha_d(\mathbf{i}_n)]),$$

where $\mathbf{i}_n$ *denotes the* n *length abbrevation of* $\mathbf{i}$*.*

For the next definition we need to define a monotonically decreasing function $Z_{\mathbf{i}\wedge\mathbf{j}}(r)$. As before, we denote the common part of $\mathbf{i}$ and $\mathbf{j}$ with $\mathbf{i} \wedge \mathbf{j}$, and write $\alpha_0(\mathbf{i} \wedge \mathbf{j}) := \infty$, $\alpha_{d+1}(\mathbf{i} \wedge \mathbf{j}) = 0$. Let $Z_{\mathbf{i}\wedge\mathbf{j}}(r) : [0, \infty) \to [0, 1]$ be defined as

$$Z_{\mathbf{i}\wedge\mathbf{j}}(r) := \prod_{k=1}^{d} \frac{\min\{r, \alpha_k(\mathbf{i} \wedge \mathbf{j})\}}{\alpha_k(\mathbf{i} \wedge \mathbf{j})}$$

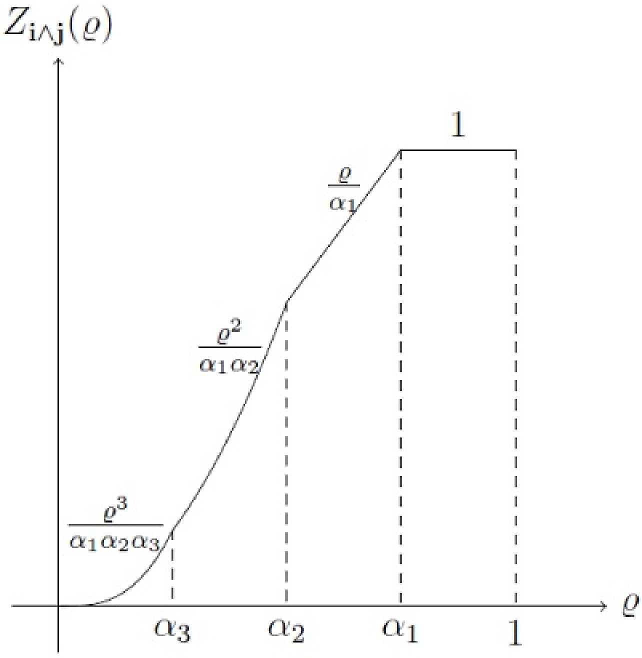

Figure 5.2: The function $Z_{\mathbf{i}\wedge\mathbf{j}}$, with $\alpha_k = \alpha_k(\mathbf{i} \wedge \mathbf{j})$

Figure 5.3 from [10] gives a better understanding about this function. By writing I_k for the interval between $\alpha_k(\mathbf{i} \wedge \mathbf{j})$ and $\alpha_{k+1}(\mathbf{i} \wedge \mathbf{j})$ we observe

$$Z_{\mathbf{i}\wedge\mathbf{j}}(r) = \sum_{k=1}^{d} \frac{r^k}{\alpha_1(\mathbf{i} \wedge \mathbf{j}) \cdot \ldots \cdot \mathbf{i} \wedge \mathbf{j}} \mathbb{1}_{J_k} + \mathbb{1}_{J_0}$$

Definition 5.3.2. *(Self-affine transversality condition) There exists a $C > 0$ constant, such that*

$$\mathcal{M}\{u \in U : |\Pi^u(\mathbf{i}) - \Pi^u(\mathbf{j})| < r\} < C \cdot Z_{\mathbf{i}\wedge\mathbf{j}}(r)$$

holds for every $\mathbf{i}, \mathbf{j} \in \Omega$, $\mathbf{i} \neq \mathbf{j}$, and this C is independent of $\mathbf{i}, \mathbf{j}$.

In almost all cases the Self-affine Hölder Condition holds, and can be checked easily. On the other hand the Self-affine Transversality Condition does not hold in most cases, and it is hard to see if this condition holds.

With these definitions we can say the following theorems, what will help us proof the main theorems. One can see, that by substituting U with D^∞, and Ω with Σ we observe the main theorems with two extra conditions, the Self-affine Hölder nodition and the Self-affine transversality condition.

Theorem 5.3.3. *Let U be a finite metric space, and $\mathcal{M}$ be a finite Borel measure on U. Let Π and Ω be as we defined before. Consider the IFS $\{T_i + c_i\}_{i=1}^k$. If the Self-affine Hölder condition and the Self-affine transversality condition hold then we have*

1. $\dim_H(\Pi^u(\Omega)) = \min\{d, d_\Omega(T_1, ...T_k)\}$,
2. *If* $d_\Omega(T_1, ...T_k) > d$, *then* $\mathcal{L}^d(\Pi^u(\Omega)) > 0$,

for almost all $u \in U$ in the sense of $\mathcal{M}$.

In the theorem above we wrote $d_\Omega(T_1, ..., T_k)$ for the affinity dimension on the Ω space of finite words. Hence,

$$s_\Omega(T_1, ..., T_k) = \inf\{s : \mathcal{N}^s(\Omega) = 0\} = \sup\{s : \mathcal{N}^s(\Omega) = \infty\},$$

where $\mathcal{N}^s$ denotes the measure we introduced in Section 3.2.

Theorem 5.3.4. *Let U be a finite metric space, and $\mathcal{M}$ be a finite Borel measure on U. Let Π and Ω be as we defined before. Consider the IFS $\{T_i+c_i\}_{i=1}^k$, and a ν ergodic measure on Ω. If the Self-affine Hölder condition and Self-affine transversality condition hold, then we have*

1. $\dim \Pi^u_*(\nu) = \min\{d, D(\nu)\}$,
2. *If* $h_\nu + \lambda_1 + \lambda_2 + ... + \lambda_d > 0$, *then* $\Pi^u_*(\nu) \ll \mathcal{L}^d$,

for almost all $u \in U$ in the sense of $\mathcal{M}$.

The next proposition gives the lower bound to these theorems.

Proposition 5.3.1. *Let U be a finite metric space with a finite Borel measure $\mathcal{M}$. Let Π and Ω be as we defined before. Consider the IFS $\{T_i + c_i\}_{i=1}^k$, and suppose that the Self-affine transversality condition holds. Let μ be a Radon measure on Ω, and choose such an $s > 0$ number, that for some $c > 0$*

$$\mu([\omega]) \le c\phi^s(A_\omega)$$

holds for all finite $\omega \in \Sigma$. Then, we get for $\mathcal{M}$ - almost all $u \in U$

1. *If* $s \le d$, *then* $\dim_H(\Pi^u_*(\mu)) \ge s$

2. *Otherwise, if* $s < d$, *then* $\Pi^u_*(\mu) \ll \mathcal{L}^d$

We will need the following Lemma in connection with the measure $\mathcal{N}^s$ for technical reasons.

Lemma 5.3.1. *If* $\mathcal{N}^s = \infty$ *for a given* s*, then there exists a finite measure* μ *supported on* Ω*, that*

$$\mu(\omega) \le c\, \phi^s(T_\omega)$$

holds for all $\omega \in \Sigma$ *finite word with some constant* c*.*

Now we can proof Theorem 5.3.3. To do that we will use Proposition 5.3.1, and the Lemma above.

Proof. First we proof part 1 of the theorem. The definition of the singular value function grants us the upper bound

$$\dim_H(\Pi^u(\Omega)) \le \min\{d, d_\Omega(T_1, ..., T_k)\}$$

To complete the proof of this part of the theorem we need the lower bound. Fix a given $s < \min\{d, d_\Omega(T_1, ..., T_k)\}$, by definition we have $\mathcal{N}^s(\Omega) = \infty$, where $\mathcal{N}^s$ denotes the measure what we used in Section 3.2 to define the affinity dimension. By using Lemma 5.3.1 we get that there exists a measure μ supported on Ω such that for some c constant

$$\mu([\omega]) \le c\, \phi^s(A_\omega)$$

By applying Proposition 5.3.1 we observe $\dim_H(\Pi^u(\Omega)) \ge s$, thus part 1 of the theorem is proofed.

Now we proof part 2. We are supposing that $d < s_\Omega(T_1, ..., T_k)$ so we can choose an s which satisfies

$$d < s < d_\Omega(T_1, ..., T_k)$$

The definition of $d_\Omega(T_1, ..., T_k)$ let us use Lemma 5.3.1 again, so there exists a measure μ supported on Ω and a c constant such that

$$\mu([\omega]) \leq c\, \phi^s(A_\omega)$$

holds for every $\omega \in \Sigma$ finite words. By using part 2 of Proposition 5.3.1 we observe $\Pi^u_*(\mu) \ll \mathcal{L}^d$ for $\mathcal{M}$-almost all $u \in U$, thus $\mathcal{L}^d(\Pi^u(\Omega)) > 0$. □

The proof of Theorem 5.3.4 follows. In the proof we write $\mathbf{i}_n$ for the first n terms of $\mathbf{i}$.

Proof. We begin with the proof of part 2 of the theorem. From the definition of $Z_{\mathbf{i}\wedge\mathbf{j}}(r)$ and the singular value function we have

$$Z_{\mathbf{i}\wedge\mathbf{j}}(r) \leq \frac{r^d}{\phi^d(A_{\mathbf{i}\wedge\mathbf{j}})}$$

for disjoint $\mathbf{i}, \mathbf{j} \in \Omega$. We assumed that the Self-affine transversality condition holds, hence we have

$$\mathcal{M}\{u \in U : |\Pi^u(\mathbf{i}) - \Pi^u(\mathbf{j})| \leq r\} \leq C \frac{r^d}{\phi^d(A_{\mathbf{i}\wedge\mathbf{j}})}$$

For ν-almost every $\mathbf{i} \in \Omega$ we also observe

$$\lim_{n\to\infty} \frac{1}{n} \log \nu([\mathbf{i}_n]) = -h_\nu \tag{5.1}$$

by the Shannon-McMillian-Breiman Theorem [13], and

$$\lim_{n\to\infty} \frac{1}{n} \log \alpha_k(A_{\mathbf{i}_n}) = \lambda_k(\nu)$$

by the proof of Oseledec's theorem [8]. It follows that

$$\lim_{n\to\infty} (\alpha_1(A_{\mathbf{i}_n}) \cdot ... \cdot \alpha_1(A_{\mathbf{i}_n}))^{\frac{1}{n}} = e^{\lambda_1(\nu)+...+\lambda_d(\nu)} \tag{5.2}$$

for almost all $\mathbf{i} \in \Omega$ in the sense of ν.

To show the absolute continuity of $\Pi^u_*(\nu)$ it is enough to show that for every $\varepsilon > 0$ the measure $\Pi^u_*(\nu_\varepsilon)$ is absolutely continuous, where ν_ε is the restriction of ν to set with measure bigger than $1-\varepsilon$. It follows from Egorov's theorem that for every $\varepsilon > 0$ there exists an $X_\varepsilon \subset \Omega, \quad \nu(x_\varepsilon) > 1-\varepsilon$ subset, and in (5.1) and (5.2) the convergence is uniform. Let ν_ε be the restriction of ν to the set X_ε. We supposed that $h_\nu + \lambda_1(\nu) + ... + \lambda_d(\nu) > 0$ thus we can choose a $\delta > 0$ such that

$$h_\nu - \delta > -(\lambda_1(\nu) + ... + \lambda_d(\nu)) + \delta$$

Let $N \in \mathbb{N}$ be such that for $n \geq N$, $\mathbf{i} \in X_\varepsilon$ the following inequalities hold

$$\nu(\mathbf{i}_n) \leq e^{n(-h_n u + \delta)}$$
$$\phi^d(A_{\mathbf{i}_n}) \geq e^{n(\lambda_1(\nu)+...+\lambda_d(\nu)-\delta)}$$

There exists such an N since the convergence is uniform in (5.1) and (5.2). By that, there exist a $d < s$ that for all $\omega \in \Sigma$ finite words $\nu_\varepsilon([\omega]) < c\phi^s(A_\omega)$ holds for some c constant. By using Propostition 5.3.1 we deduce that $\Pi_*^u(\nu_\varepsilon) \ll \mathcal{L}^d$. It holds for all $\varepsilon > 0$ so the proof of part 2 is complete.

Next we proof part 1 of the theorem. We give the lower bound first, and then the upper bound to show the equality. Assume that $D(\nu) < d$, and let $k = \max\{i : 0 < h_\nu + \lambda_1(\nu) + ... + \lambda_i(\nu)\}$ as in the definition of the Lyapunov dimension. Since $k < D(\nu) \leq k+1$, let $k < s < D(\nu)$. Choose an $\varepsilon > 0$ such that

$$D(\nu) - s = \frac{2\varepsilon}{-\lambda_{k+1}}$$

From Lemma 5.2.1 we have

$$E_\nu(s) > -h_\nu + 2\varepsilon$$

By Egorov's Theorem for every $\delta > 0$ there exists a $H_\delta \subset \Omega$ set with $\nu(H_\delta) > 1-\delta$. Using the Shannon-McMillan-Breiman Theorem, and the definition of $E_\nu(s)$ we get that there exists such an N, that for $n \geq N$

$$\nu([\mathbf{i}_n]) \leq e^{n(-h_\nu+\varepsilon)} < e^{nE_\nu(s)} < \phi^s(A_{\mathbf{i}_n})$$

for every $\mathbf{i} \in H_\delta$. For all $\mathbf{i} \in H_\delta$ we observe

$$\nu([\mathbf{i}_n]) < \phi^s(A_{\mathbf{i}_n})$$

Let ν_δ be the restriction of ν to H_δ. Using Proposition 5.3.1 we get

$$\dim_H(\Pi_*^u(\nu)) \geq \dim_H(\Pi_*^u(\nu_\delta)) \geq s$$

for $\mathcal{M}$-almost all $u \in U$. Thus we get the lower estimate in part 1, beacause $s < D(\nu)$ was arbitrary.

We need the upper bound in part 1, to get that let $D(\nu) < d$. Fix a given $u \in U$, we will give the upper bound for $\Lambda := \Pi^u(\Omega)$. It is enough to proof that

$$\dim_H(\Pi_*^u(\nu)) \leq s'$$

holds for all $s' > D(\nu)$. We choose an $\varepsilon > 0$ what satisfies

$$s' > \frac{h_\nu + \lambda_1(\nu) + ... + \lambda_k(\nu)}{-\lambda_{k+1} + \varepsilon} + k + \frac{(k+1)\varepsilon}{-\lambda_{k+1} + \varepsilon} \tag{5.3}$$

where k is the number defined in the definition of the Lyapunov dimension, thus $k = \max\{i : 0 < h_\nu + \lambda_1(\nu) + ... + \lambda_i(\nu)\}$. We assumed the Self-affine Hölder condition so we have a $K > 0$ such that $\Pi^u(\mathbf{i}_n)$ can be covered by a rectangular box with side lengths $K\alpha_1(\mathbf{i}_n), ..., K\alpha_d(\mathbf{i}_n)$, for all $\mathbf{i} \in \Omega$, $n \in \mathbb{N}$. Denote this box with $B_{\mathbf{i}_n}$. For all $\mathbf{i} \in \Omega$, $n \in \mathbb{N}$ we divide $B_{\mathbf{i}_n}$ into $N(\mathbf{i}_n)$ boxes with sides $l(\mathbf{i}_n) \in \mathbb{R}^d$, the n^{th} term of $l(\mathbf{i}_n)$ is the length of the n^{th} side.

$$N(\mathbf{i}_n) := \frac{\alpha_1(\mathbf{i}_n)...\alpha_k(\mathbf{i}_n)}{\alpha_{k+1}^k(\mathbf{i}_n)}$$

$$l(\mathbf{i}_n) := (\underbrace{\alpha_{k+1}(\mathbf{i}_n), ..., \alpha_{k+1}(\mathbf{i}_n)}_{k+1}, \alpha_{k+2}(\mathbf{i}_n), ..., \alpha_d(\mathbf{i}_n))$$

Write $P_n(\mathbf{i})$ for the set containing $\Pi^u(\mathbf{i})$, and $Q_n(\mathbf{i}) \subset \Omega$ for the set corresponding to $P_n(\mathbf{i})$, thus

$$Q_n(\mathbf{i}) := \{\mathbf{j} \in \Omega : \mathbf{j} \in [\mathbf{i}_n], \quad \Pi^u(\mathbf{j}) \in P_n(\mathbf{i})\}.$$

Define the set

$$A_\varepsilon^n := \left\{\mathbf{i} \in \Omega : \nu(Q_n(\mathbf{i})) \geq \varepsilon \, \frac{\nu(\mathbf{i}_n)}{N(\mathbf{i}_n)}\right\}$$

From this definition we have

$$\nu((A_\varepsilon^n)^c) = \sum_{|\omega|=n} \nu([\omega]) \cap (A_\varepsilon^n)^c) \leq \sum_{|\omega|=n} N(\omega)\, \varepsilon\, \frac{\nu([\omega])}{N(\omega)} \leq \varepsilon.$$

By that we obtain that $\nu(A_\varepsilon) > 1 - \varepsilon$, where

$$A_\varepsilon := \bigcap_{i=1}^{\infty} \bigcup_{n=i}^{\infty} A_\varepsilon^n$$

By Using Egorov's theorem again we have that there exist a $G_\varepsilon \subset A_\varepsilon$ with $\nu(G_\varepsilon) > 1 - 2\varepsilon$. As before we have

$$\lim_{n\to\infty} \log \nu([\mathbf{i}_n]) = -h_\nu$$
$$\frac{1}{n} \log \alpha_l(\mathbf{i}_n) \to \lambda_l(\nu),$$

with uniform convergence for all $1 \leq l \leq d, \quad \mathbf{i} \in G_\varepsilon$. These implie that there exists a sufficiently large n for all $\mathbf{i} \in G_\varepsilon$ that the following conditions hold:

- $\nu(Q_n(\mathbf{i})) > \varepsilon\, \frac{\nu(\mathbf{i}_n)}{N(\mathbf{i_n})}$
- $e^{n(-h_\nu - \varepsilon)} < \nu(\mathbf{i}_n) < e^{n(-h_\nu + \varepsilon)}$
- $e^{n(\lambda_l(\nu) - \varepsilon)} < \alpha_l(\mathbf{i}_n) < e^{n(\lambda_l(\nu) + \varepsilon)}$ for all $1 \leq l \leq d$
- $\frac{\log \varepsilon}{\log \alpha_{k+1}(\mathbf{i}_n)} < \varepsilon$

Choose such an n number, and fix $\mathbf{i} \in G_\varepsilon$. To complete the proof we want to use Frostman's Lemma to the measure $\Pi_*^u(\nu)$, to do that we need the following estimate

$$\begin{aligned} R_n(\mathbf{i}) &:= \frac{\log \Pi_*^u(\nu)\, [B(\Pi^u(\mathbf{i}), \alpha_{k+1}(\mathbf{i}_n))]}{\log \alpha_{k+1}(\mathbf{i}_n)} \leq \frac{\log \nu(Q_n(\mathbf{i}))}{\log \alpha_{k+1}(\mathbf{i}_n)} \\ &\leq \frac{\log \varepsilon}{\log \alpha_{k+1}(\mathbf{i}_n)} + \frac{\log \nu(\mathbf{i}_n) - \log N(\mathbf{i}_n)}{\log \alpha_{k+1}(\mathbf{i}_n)} \\ &\leq \varepsilon + \frac{\log N(\mathbf{i}_n) - \log \nu(\mathbf{i}_n)}{-\log \alpha_{k+1}(\mathbf{i}_n)} \\ &\leq \varepsilon + \frac{n(\lambda_1(\nu) + ... + \lambda_k(\nu) + k\varepsilon - k\lambda_{k+1}(\nu) + k\varepsilon) + n(h_\nu + \varepsilon)}{-n(\lambda_{k+1}(\nu) + -\varepsilon)} \\ &= \frac{h_\nu + \lambda_1(\nu) + ... + \lambda_k(\nu)}{-\lambda_{k+1} + \varepsilon} + k + \frac{(k+1)\varepsilon}{-\lambda_{k+1} + \varepsilon} \end{aligned}$$

We get that

$$\liminf_{n\to\infty} R_n(\mathbf{i}) < s'$$

holds for all $\mathbf{i} \in G_\varepsilon$, since we assumed that ε satisfies (5.3). We constructed the sets G_ε in such a way, that $G_{\varepsilon_2} \subset G_{\varepsilon_1}$ if $\varepsilon_1 < \varepsilon_2$ thus the estimate above holds for almost all $\mathbf{i} \in \Omega$ in the sense of ν. Now we can use Frostman's Lemma to get $\dim_H(\nu) \leq s'$, since s' was an $s' > D(\nu)$ arnitrary number it completes the proof. □

The main theorems are the immediate consequences of Theorems 5.3.3 and 5.3.4, if the Self-affine transversality condition holds. That is why we need the next lemma.

Lemma 5.3.2. *For every $\mathbf{y} \in D^\infty$, the Self-affine transversality condition holds for $\Pi^{\mathbf{y}}$. Thus, there exists a $C > 0$ constant, such that*

$$\mathbb{P}\{\mathbf{y} \in D^\infty : |\Pi^{\mathbf{y}}(\mathbf{i}) - \Pi^{\mathbf{y}}(\mathbf{j})| < r\} < C \cdot Z_{\mathbf{i}\wedge\mathbf{j}}(r)$$

holds for all $\mathbf{i}, \mathbf{j} \in \Sigma$, $\mathbf{i} \neq \mathbf{j}$.

Now we have all we needed for the proof of Theorem 5.3.1, and Theorem 5.3.2 .

Proof. The proof of Theorem 5.3.1 immediately follows from Theorem 5.3.3 and Lemma 5.3.2. □

Proof. The proof of Theorem 5.3.2 immediately follows from Theorem 5.3.4 and Lemma 5.3.2. □

6. The Ledrappier - Young formula

In this section we are going to express the Hausdorff dimension of the invariant measure of a self-affine fractal with the dimension of it's projected measure, in $\mathbb{R}^2$. Thus if the invariant measure is ν, and it's projection to the axis x is ν_x, then we will express $\dim_H \nu$ with $\dim_H \nu_x$. To do this we need to introduce the so called projection entropy. Using that we can proof the theorem of Feng and Hu. The results of this section are due to Feng, Hu [4]. The way of the presentation follows the line [10].

6.1 The main theorem

Consider a self-affine iterated function system on $\mathbb{R}^2$ of the following kind:

$$\mathcal{F}_{\beta,\tau} := \left\{ f_i(x) := \begin{bmatrix} \beta_i & 0 \\ 0 & \tau_i \end{bmatrix} \cdot x + t_i \right\}, \text{ where } 0 < \beta_i, \tau_i < 1 \tag{6.1}$$

Let μ be an ergodic measure on the symbolic space Σ, and Π be the natural projection from Σ to the plane. Write ν and ν_x for the push-down measures of μ by Π and $proj_x \circ \Pi$ respectively, hence

$$\nu := \Pi_* \mu \text{ , and } \nu_x := (proj_x \circ \Pi)_* \mu,$$

where $proj_x$ is the projection to the axis s. The linear part of the functions are given by diagonal matrices, so the coordinate axes will be the eigenspaces.

Theorem 6.1.1. *(Feng, Hu [4]) Suppuse that the IFS $\mathcal{F}_{\beta,\tau}$ we defined above satisfies the strong separation condition, and x is the weak contraction direction, thus $\lambda_1(\mu) = \lambda_x(\mu) \geq \lambda_y(\mu) = \lambda_2(\mu)$. Then*

$$\dim_H(\nu) = \frac{h_\mu}{-\lambda_2(\mu)} + \left(1 - \frac{\lambda_1(\mu)}{\lambda_2(\mu)}\right) \cdot \dim_H(\nu_x)$$

The following corollary is the immediate consequence of the theorem.

Corollary 6.1.1. *By assuming the same as in theorem 6.1.1 we observe*

$$D(\nu) = \dim_H(\nu) \Leftrightarrow \dim_H(\nu_x) = \min\left\{1, \frac{h_\mu}{-\lambda_1(\mu)}\right\}$$

This formula makes the calculation of the Hausdorff dimension an easy task, if we know $\dim_H(\nu_x)$, as it is shown in the following example.

Consider the following IFS on $[0,1]^2$

$$\left\{ f_1(x) = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} \cdot x + \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad f_2(x) = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} \cdot x + \begin{bmatrix} 1/2 \\ 2/3 \end{bmatrix} \right\}$$

You can see how the first cylinder of this system looks like on figure 6.1. To calculate the Hausdorff dimension of this fractal we will use Theorem 6.1.1, since all the sufficiant assumptions are hold in this case. We have

$$h_\mu = \log 2, \quad \lambda_1(\mu) = \log\frac{1}{2} = -\log 2, \quad \lambda_2(\mu) = \log\frac{1}{3} = -\log 3$$

where μ is the probability vector $\left(\frac{1}{2}, \frac{1}{2}\right)$. By construction, the projection to the x axis is a disjunct cover of $[0,1]$, thus

$$\nu_x \equiv \mathcal{L}^1 \Longrightarrow \dim_H(\nu_x) = 1$$

By using the formula in Theorem 6.1.1 we obtain

$$\dim_H(\nu) = \frac{\log 2}{\log 3} + \left(1 - \frac{\log 2}{\log 3}\right) \cdot 1 = 1$$

It's a good lower bound to the Hausdorff dimension of the attractor, to get the upper bound we need the affinity dimension. Using the formula we observed in the example in Chapter 3

$$d(T_1, T_2) = s, \text{ where } \frac{1}{2}\left(\frac{1}{3}\right)^{s-1} + \frac{1}{2}\left(\frac{1}{3}\right)^{s-1} = 1$$

We obtained that the affinity dimension is also 1, thus $\dim_H(\Lambda) = 1$, where Λ is the attractor of the IFS.

In the example it was really easy to get the Hausdorff dimension of the measure ν, but only because the projection to the weak contraction direction was trivial (no overlapping). However in the case of overlapping we can use the next theorem. The ν in the following corresponds to ν_x.

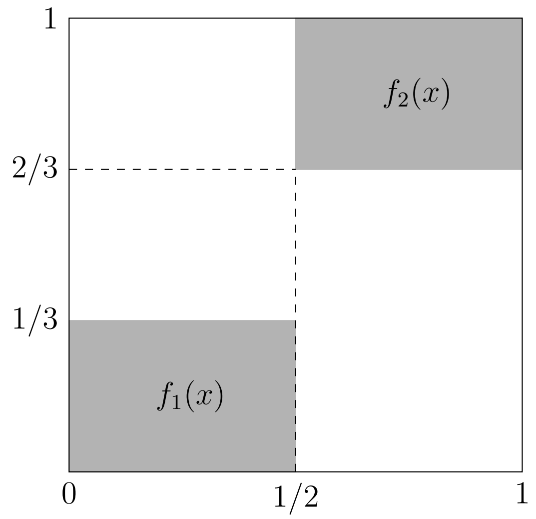

Figure 6.1: The first cylinder of the fractal in the example

Theorem 6.1.2. *(Feng, Hu [4]) Consider the IFS* $\{S_i(x) = r_i x + t_i\}_{i=1}^m$ *with* $0 < r_i < 1$. *Let* μ *be an ergodic measure on* Σ, *and* $\nu := \Pi_* \mu$ *be it's push-down measure by the natural projection* Π. *Write* λ_ν *for the Lyapunov exponent of* ν. *Then we have*

$$d_\nu(x) = \frac{h_\Pi}{-\lambda_\nu}$$

for almost all x *in the sense of* ν.

In the Theorem above we used the notation $d_n u(x)$ for the local dimension of ν at x, and h_Π for the projection entropy. The definition of h_Π will be given in the following section.

6.2 The projection entropy

Consider the following IFS on the line

$$\{S_i(x) = r_i x + t_i\}_{i=1}^k \text{ , with } 0 < r_i < 1$$

To make the formulas shorter we write $S_{i_1 \ldots i_n} = S_{i_1} \circ \ldots \circ S_{i_n}$, and $r_{i_1 \ldots i_n} = r_{i_1} \cdot \ldots \cdot r_{i_n}$. Let Λ be the attractor, and Σ be the symbolic space. As usual we write Π for the natural projection from the symbolic space to the line. Further we will use the following notations:

- $\gamma := \mathcal{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$
- Write $\mathcal{M}_\sigma(\Sigma)$ for the set of invariant Borel probability measures on Σ
- Write $E_\sigma(\Sigma)$ for the set of Ergodic measures on Σ
- $\mathcal{I}$ is the set of the σ invariant measurable subsets of Σ, hence $\mathcal{I} := \{A \subset \mathcal{B}(\Sigma) : \sigma^{-1}A = A\}$
- Let $\mathcal{P}$ be the partition of Σ by the first digits
- Fix $\mu \in E_\sigma(\Sigma)$, $\nu := \Pi_*\mu$ is the push-down measure of this μ
- We write λ_ν for the Lyapunov exponent of ν

Definition 6.2.1. *(Projection entropy)*

$$h_\Pi := h_{\Pi,\mu}(\sigma) := H_\mu(\mathcal{P}|\sigma^{-1}\Pi^{-1}\gamma) - H_\mu(\mathcal{P}|\Pi^{-1}\gamma)$$

Before the proof of Theorem 6.1.2, we represent some properties of the projection entropy. Write $\mathcal{D}_n$ for the prtition generated by the dyadic intervals $\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right]$, $\quad i = 1, 2, ..., 2^n$. Using the properties of the conditional entropy we observe

$$h_\Pi = \lim_{n\to\infty} H(\mathcal{P}|\sigma^{-1}\Pi^{-1}\mathcal{D}_n) - H(\mathcal{P}|\Pi^{-1}\mathcal{D}_n) \tag{6.2}$$

$$\begin{aligned} H(\mathcal{P}|\sigma^{-1}\Pi^{-1}\mathcal{D}_n) &- H(\mathcal{P}|\Pi^{-1}\mathcal{D}_n) \\ &= H(\mathcal{P} \vee \sigma^{-1}\Pi^{-1}\mathcal{D}_n) - H(\mathcal{P} \vee \Pi^{-1}\mathcal{D}_n) \end{aligned}$$

Lemma 6.2.1. *The following statements hold for the projection entropy*

1. $0 \le h_\Pi \le h_\mu$
2. $\mathcal{P} \vee \sigma^{-1}\Pi^{-1}\gamma = \mathcal{P} \vee \Pi^{-1}\gamma$

We only proof the second part of the lemma.

Proof. It is easy to see that

$$\mathcal{P} \vee \sigma^{-1}\Pi^{-1}\gamma = \bigcup_{i=1}^{k} \left\{ [i] \cap \sigma^{-1}\Pi^{-1}A_i \right\}$$

for some $A_i \in \gamma$ Borel sets. Since we have

$$[i] \cap \sigma^{-1}\Pi^{-1}A_i = [i] \cap \Pi^{-1}S_i(A_i)$$

we obtain $\mathcal{P} \vee \sigma^{-1}\Pi^{-1}\gamma \subset \mathcal{P} \vee \Pi^{-1}\gamma$, because $S_i(A_i) \in \gamma$. The other direction can be proofed a similar way. We can write the prartition in the following form again

$$\mathcal{P} \vee \Pi^{-1}\gamma = \bigcup_{i=1}^{k} \left\{ [i] \cap \Pi^{-1}A_i \right\}$$

for some $A_i \in \gamma$ Borel sets. Since $S_i^{-1}(A_i) \in \gamma$, the proof is complete by using

$$[i] \cap \Pi^{-1}A_i = [i] \cap \Pi^{-1}S_i^{-1}(A_i)$$

□

6.3 The proof of Theorem 6.1.2

We will need the following proposition, and the lemma after it for the proof.

Proposition 6.3.1. *(Mane's book Corollary 1.6) Consider a $\mu \in \mathcal{M}_\sigma(\Sigma)$ invariant measure, and an*
$F_k \in L^1(\Sigma, \mu)$ sequence which converges to $F \in L^1(\Sigma, \mu)$ almost everywhere in L^1. Then we have almost everywhere

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} F_{n-i}(\sigma^i(x)) = \mathbb{E}_\mu \left[F | \mathcal{I} \right](x)$$

where the convergence is in the sense of the convergence in L^1.

Lemma 6.3.1. *Let $\Phi : \Sigma \to \Lambda$ be a continuous map, and ξ, η be finite partitions of Σ with $H(\xi) < \infty$. We write*

$$B^{\Phi}(\mathbf{i}, r) := \Phi^{-1}B(\Phi(\mathbf{i}), r)$$
$$g(\mathbf{i}) := -\inf_{0<r} \log \frac{\mu(B^{\Phi}(\mathbf{i}, r) \cap \eta(\mathbf{i}) \cap \xi(\mathbf{i}))}{\mu(B^{\Phi}(\mathbf{i}, r) \cap \eta(\mathbf{i}))}$$

Then we obtain:

1. $\lim_{r\to 0} \log \frac{\mu(B^{\Phi}(\mathbf{i},r)\cap\eta(\mathbf{i})\cap\xi(\mathbf{i}))}{\mu(B^{\Phi}(\mathbf{i},r)\cap\eta(\mathbf{i}))} = -I_{\mu}(\xi|\hat{\eta} \wedge \Phi^{-1}\gamma)(\mathbf{i})$ *for almost all $\mathbf{i} \in \Sigma$ in the sense of μ, where $\hat{\eta}$ is the σ-algebra generated by η.*

2. $g \in L^1(\Sigma, \mathcal{B}(\Sigma), \mu)$*, where $\mathcal{B}(\Sigma)$ is the Borel σ-algebra on Σ.*

From this Lemma we conclude

$$\lim_{r\to 0} \log \frac{\mu(B^{\Phi}(\mathbf{i}, r) \cap \eta(\mathbf{i}) \cap \xi(\mathbf{i}))}{\mu(B^{\Phi}(\mathbf{i}, r) \cap \eta(\mathbf{i}))} = -I_{\mu}(\xi|\hat{\eta} \wedge \Phi^{-1}\gamma)(\mathbf{i})$$

hold μ-almost everywhere with the convergence in L^1.

The following Theorem will come handy to give our proof about the local dimension instead of the Hausdorff dimension. We write $d_{\mu}(x)$ for the local dimension, and $\underline{d}$ for the lower local dimension of μ at x.

Theorem 6.3.1. *Let μ be a mass distribution on $\mathbb{R}^d$, thus μ is a Borel measure with $0 < \mu(\mathbb{R}^d) < \infty$. Then for the Hausdorff dimension of μ we have*

$$\begin{aligned} \dim_H(\mu) &= \operatorname*{ess\,sup}_x \underline{d}_{\mu}(x) \\ &= \inf\{\alpha : \underline{d}_{\mu}(x) \leq \alpha \text{ , for } \mu\text{-almost all } x\} \end{aligned}$$

For some $E \subset \mathbb{R}^d$, if $\mu(E) = 1$, $d_{\mu}(x) = \alpha$ for all $x \in E$, then $\dim_H(E) = \alpha$.

The proof of Theorem 6.1.2 follows, it also makes the understanding of the projection entropy easier.

Proof. For an $\mathbf{i}$ we need the $s_k : \Sigma \to \mathbb{R}^+$ sequence of functions

$$s_0(\mathbf{i}) \equiv 1, \quad s_k(\mathbf{i}) = r_{i_1} \cdot ... \cdot r_{i_k}, \text{ if } k \geq 0$$

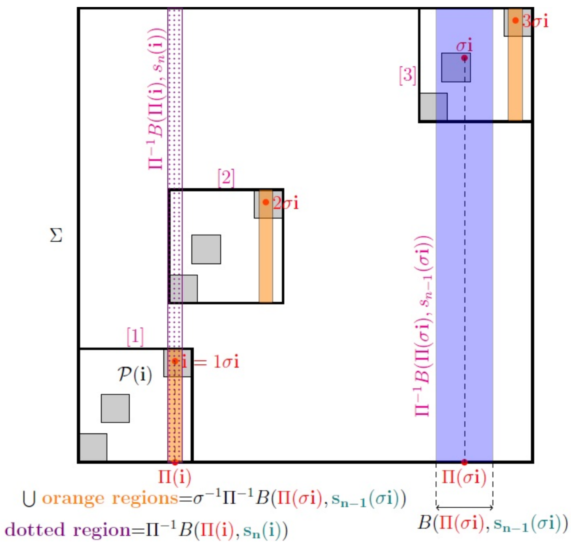

Figure 6.2: This Figure from [10] represents the sets, and their measures we use in the proof

Using Theorem 6.3.1 the proof is complete, if we show that

$$\lim_{n\to\infty} -\frac{1}{n} \log \mu\left(B^{\Pi}(\mathbf{i}, s_n(\mathbf{i}))\right) = h_{\Pi}$$

holds for μ-almost all $\mathbf{i} \in \Sigma$. For all $\mathbf{i} \in \Sigma$, and $r > 0$ we observe

$$B^{\Pi\sigma}(\mathbf{i}, r) \cap \mathcal{P}(\mathbf{i}) = B^{\Pi}(\mathbf{i}, r_{i_1} r) \cap \mathcal{P}(\mathbf{i})$$
$$\mu(B^{\Pi}(\sigma\mathbf{i}, r)) = \mu(B^{\Pi\sigma}(\mathbf{i}, r))$$

using the invariance of μ. We define the functions $H_n(\mathbf{i}), W_n(\mathbf{i})$, and $G_n(\mathbf{i})$ in the following way

$$H_n(\mathbf{i}) := \log \frac{\mu(B^{\Pi}(\mathbf{i}, s_n(\mathbf{i})))}{\mu(B^{\Pi}(\sigma\mathbf{i}, s_{n-1}(\sigma\mathbf{i})))}$$
$$W_n(\mathbf{i}) := \log \frac{\mu(B^{\Pi\sigma}(\mathbf{i}, s_{n-1}(\mathbf{i})) \cap \mathcal{P}(\mathbf{i}))}{\mu(B^{\Pi\sigma}(\mathbf{i}, s_{n-1}(\sigma\mathbf{i})))}$$
$$G_n(\mathbf{i}) := \log \frac{\mu(B^{\Pi}(\mathbf{i}, s_n(\mathbf{i})) \cap \mathcal{P}(\mathbf{i}))}{\mu(B^{\Pi}(\mathbf{i}, s_n(\mathbf{i})))}$$

By using the notation on Figure 6.3] we have

$$H_n(\mathbf{i}) = \log \frac{\mu(dotted)}{\mu(orange)} = \log \frac{\mu(dotted)}{\mu(blue)},$$

$$W_n(\mathbf{i}) = \log \frac{\mu(dottedorange)}{\mu(orange)}, \quad G_n(\mathbf{i}) = \log \frac{\mu(dottedorange)}{\mu(dotted)}$$

From these we conclude

$$W_n(\mathbf{i}) - G_n(\mathbf{i}) = \log \frac{\frac{\mu(dottedorange)}{\mu(orange)}}{\frac{\mu(dottedorange)}{\mu(dotted)}} = \log \frac{\mu(dotted)}{\mu(orange)} = H_n(\mathbf{i}) \tag{6.3}$$

Further in the proof we will use the telescoping series

$$\sum_{k=0}^{n-1} H_{n-k}(\sigma^k \mathbf{i}) = \log \mu(B^{\Pi}(\mathbf{i}, s_n(\mathbf{i}))) - \log \mu(B^{\Pi}(\sigma^n \mathbf{i}, 1))$$

One can see that $\lim_{n\to\infty} \frac{1}{n} \log \mu(B^{\Pi}(\sigma^n \mathbf{i}, 1)) = 0$, thus from (6.3) we obtain

$$\lim_{n\to\infty} \frac{1}{n} \log \mu(B^{\Pi}(\mathbf{i}, s_n(\mathbf{i}))) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left[W_{n-k}(\sigma^k \mathbf{i}) - G_{n-k}(\sigma^k \mathbf{i})\right] \tag{6.4}$$

By using Lemma 6.3.1 we have

$$G_n \to G := -I_\mu(\mathcal{P}|\Pi^{-1}\gamma)$$

with $\Phi = \Pi$, and

$$W_n \to W := -I_\mu(\mathcal{P}|\sigma^{-1}\Pi^{-1}\gamma)$$

with $\Phi = \Pi \circ \sigma$. We can use Proposition 6.3.1, and the ergodicity of μ to get the following two limits for almost all $\mathbf{i} \in \Sigma$ in the sense of μ

$$\frac{1}{n} \sum_{k=0}^{n-1} G_{n-k}(\sigma^k \mathbf{i}) \to \mathbb{E}_\mu \left[G|\mathcal{I}\right](\mathbf{i}) = \int G(\mathbf{i}) d\mu(\mathbf{i}) = H_\mu(\mathcal{P}|\Pi^{-1}\gamma)$$

$$\frac{1}{n} \sum_{k=0}^{n-1} W_{n-k}(\sigma^k \mathbf{i}) \to \mathbb{E}_\mu \left[W|\mathcal{I}\right](\mathbf{i}) = \int W(\mathbf{i}) d\mu(\mathbf{i}) = H_\mu(\mathcal{P}|\sigma^{-1}\Pi^{-1}\gamma)$$

From these, and (6.4) we observe

$$\lim_{n\to\infty} -\frac{1}{n} \log \mu(B^{\Pi}(\mathbf{i}, s_n(\mathbf{i}))) = H_\mu(\mathcal{P}|\Pi^{-1}\gamma) - H_\mu(\mathcal{P}|\sigma^{-1}\Pi^{-1}\gamma) = h_\Pi$$

□

7. Conclusion

This paper was set out to represent how the Ledrappier-Young formula [9] works on self-affine fractals, which are defined by diagonal maps. We also showed the main idea behind this formula, the meaning of the projection entropy, and how can we use it to express the Hausdorff dimension. Knowing some other ways of estimating the Hausdorff dimension of a given self-similar, or self-affine invariant set makes easier to understand why this formula is important, that's why we gave a summary of some of the classic results in the dimension theory of fractals.

At the start we give formulas for the Hausdorff dimension in self-affine cases, using assumptions on the IFS like the SSP, the OSC, or some other geometric conditions. We also covered the cases of overlapping cylinders, for self-similar fractals.

Since the attractor may look different due to the occurence of random errors in the translations during the iteration, we showed how can some unwanted noise of this kind modify the Hausdorff dimension.

To sum up we can say that today we can calculate the Hausdorff dimension of various fractals, even self-affine ones in some special cases. The separation of the cylinders is still required, but we can let them to overlap in an arbitrary direction (thus the overlapping of the projections of the cylinders is allowed). To give a formula for the Haudorff dimension of overlapping or non-diagonal self-affine fractals is a very active and popular research field nowadays.

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