Msc Thesis

The dimension theory of some families of non-conformal iterated function systems

Rudolf Dániel Prokaj

Supervisor:

Dr. Károly Simon

MŰEGYETEM 1782

Department of Stochastics

BUTE, Institute of Mathematics

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Chapter 1

Introduction

One of the main goals of fractal geometry is to give formulas for the dimension of self-similar and self-affine sets. There are many results in this topic, most of them revolve around iterated function systems on $\mathbb{R}^d$ consisting of conformal maps. In the first half of my thesis I will introduce a specific family of non-conformal systems, and investigate its behavior. We are going to use the theory of graph directed systems to gather tools with which we will be able to give a formula for the Hausdorff dimension of these systems.

Afterwards we turn our attention toward a really important notion that can be useful if one want to deal with self-affine IFS, namely the Furstenberg measure. First we only give a vague picture, and then show how it works in certain examples to make the understanding easier. We will present the results of Bárány, Rams and Simon [3] about lower triangular maps, and then the work of Falconer and Kempton [4] since these articles give a nice review on the actions of the forward and the backward Furstenberg measures. Finally we show the connection between the Lyapunov exponents of an IFS and the associated Furstenberg measure in a special case.

In the rest of this chapter we will introduce the most common definitions that will be used during this paper. We will use the notation of [14].

Definition 1.0.1. *We call the collection of contractive functions* $\{f_i\}_{i=1}^m$ *on* $\mathbb{R}^d$ *an* ***iterated function system****, or IFS for short.*

Write $f_{i_1\dots i_n} := f_{i_1}\dots f_{i_n}$, and let B be the closed ball on $\mathbb{R}^d$. Since for all $i = 1\dots m \quad f_i$ is a contraction, $\{\bigcup_{i_1\dots i_n} f_{i_1\dots i_n}(B)\}_{n=1}^{\infty}$ is a nested sequence of sets . Thus we can define the attractor of an IFS in the following way.

Definition 1.0.2. *We define the **attractor** Λ of the IFS $\{f_i\}_{i=1}^m$ as*

$$\Lambda := \bigcap_{n=1}^{\infty} \bigcup_{i_1\dots i_n} f_{i_1\dots i_n}(B)$$

The space $\Sigma := \{1, \dots, m\}^{\mathbb{N}}$ is called **symbolic space**, and the function $\Pi : \Sigma \to \Lambda$ is the **natural projection** defined by

$$\Pi(\mathbf{i}) := \lim_{n\to\infty} f_{\mathbf{i}|_n}(0),$$

where $\mathbf{i}|_n$ denotes the n-length word consisting of the first n terms of $\mathbf{i}$. We write Σ_n for the subset of n-length words, and Σ^* for the subspace of finite words. The cylinder defined by $\mathbf{i}|_n$ will always be denoted by $[\mathbf{i}|_n]$ through this paper.

We have seen that Π codes the points of the attractor with the words of the symbolic space. This coding is bijective if all the words $\mathbf{i}$ define a distinct point of Λ, thus if the following separation condition holds.

Definition 1.0.3. *The **Strong Separation Property (SSP)** holds for the IFS $\{f_i\}_{i=1}^m$ with attractor Λ, if*

$$f_i(\Lambda) \cap f_j(\Lambda) = \emptyset, \text{ for all } i \neq j$$

The following separation condition is also frequently used, and a bit less strict than the SSP.

Definition 1.0.4. *The **Open Set Condition (OSC)** holds for the IFS $\{f_i\}_{i=1}^m$ on $\mathbb{R}^d$ with attractor Λ if there exists a non-empty open set $V \subset \mathbb{R}^d$ such that*

1. *$f_i(V) \subset V$, for all $i = 1, \dots, k$*
2. *$f_i(V) \cap f_j(V) = \emptyset$, for all $i \neq j$*

Most of the fractals on $\mathbb{R}^d$ are zero measure sets with respect to the d-dimensional Lebesgue measure. For this reason we need to measure the "velocity" of a fractal in a different way. We recall here only the most common fractal dimensions.

Definition 1.0.5. *Let $F \subset \mathbb{R}^d$ be a non-empty bounded set, and write $N_\delta(F)$ for the smallest number of δ diameter sets which can cover F. Then the **lower and upper box dimensions** of F are*

$$\underline{\dim}_B(F) := \liminf_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$
$$\overline{\dim}_B(F) := \limsup_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$$

*If the limits are equal, then $\dim_B(F) := \underline{\dim}_B(F) = \overline{\dim}_B(F)$ is the **box counting dimension** of F.*

Definition 1.0.6. *Let $F \subset \mathbb{R}^d$ be a non-empty bounded set. We write $\mathcal{H}^t(F)$ for the t-dimensional Hausdorff measure of F, thus*

$$\mathcal{H}^t(F) := \lim_{\delta \to 0} \left(\inf \left\{ \sum_{i=1}^{\infty} |A_i|^t : F \subset \bigcup_{i=1}^{\infty} A_i,\ |A_i| \leq \delta \right\} \right)$$

*Then the **Hausdorff dimension** of F is the $t = \dim_H(F)$ number where the t-dimensional Hausdorff measure of F drops from infinity to 0.*

$$\dim_H(F) := \inf \left\{ t : \mathcal{H}^t(F) = 0 \right\} = \sup \left\{ t : \mathcal{H}^t(F) = \infty \right\}$$

Definition 1.0.7. *Consider the IFS $\{S_i\}_{i=1}^m$ with ratios $0 < r_i < 1$. Then the **similarity dimension** of the IFS is the unique $s = \dim_s(F)$ solution of*

$$r_1^s + \ldots + r_m^s = 1$$

In higher dimensions $d \geq 2$ the linear part of each function f_i is a matrix T_i which acts on $\mathbb{R}^d$. The **singular values** of a map T are the square roots of the eigenvalues of T^*T where T^* is the transpose of T, and we will write

$\alpha_1(T) > \alpha_2(T) > \cdots > \alpha_d(T)$ for them. The **singular value function** $\phi^s(T)$ is defined by the singular values of T in the following way

$$\phi^s(T) := \begin{cases} \alpha_1 \dots \alpha_{m-1}\alpha_m^{s-m+1}, & 0 \leq s \leq d \\ (\alpha_1\alpha_2 \dots \alpha_d)^{s/n} = \det(T)^{s/n}, & s > n, \end{cases} \tag{1.1}$$

where m is the smallest integer which is bigger than or equal to s. Let $\mathcal{T} = (T_1, \dots, T_m)$ and $s \geq 0$, then the **topological pressure function** is

$$P(\mathcal{T}, s) := \lim_{n\to\infty} \frac{1}{n} \log \left(\sum_{|\mathbf{i}|=n} \phi^s(T_{\mathbf{i}}) \right).$$

Definition 1.0.8. *The **affinity dimension** of the attractor Λ of the self-affine IFS $\{f_i = T_i + t_i\}_{i=1}^m$ is the unique zero of $P(\mathcal{T}, s)$, and we denote it by $\dim_A \Lambda$.*

We can say that the Hausdorff dimension uses the most economic cover of the set, and the affinity dimension uses the most natural one. Thus for any set $F \subset \mathbb{R}^d$ we always have

$$\dim_H F \leq \underline{\dim}_B(F) \leq \overline{\dim}_B(F) \leq \dim_A(F).$$

The measure μ is **self-similar** on Σ if $\mu(\mathbf{i}) = \sum_{j=1}^m p_j \mu(\sigma \mathbf{i})$ for a probability vector $\mathbf{p} = (p_1, \dots, p_m)$ and for each $\mathbf{i} \in \Sigma$, where σ is the left-shift on the symbolic space. The push-forward measure $\nu := \Pi_* \mu$ is a self-affine measure on Λ since by definition

$$\nu(\Lambda) = \sum_{i=1}^m p_i \nu(f_i \Lambda).$$

There are also several notions for the dimension of self-affine measures.

Definition 1.0.9. *The **Hausdorff dimension** of a measure μ is*

$$\dim_H \mu := \inf\{\dim_H E : \mu(E) = 1\}.$$

Definition 1.0.10. *Write $B(x,r)$ for the ball of radius r centered at x. The* ***local dimension*** *of μ at x is given by*

$$\dim_{loc}(\mu, x) := \lim_{\delta \to 0} \frac{\log \mu(B(x,\delta))}{\log \delta},$$

provided that the limit exists.

For a given ergodic measure μ write h_μ for its entropy, and $\lambda_1(\mu) \geq \lambda_2(\mu) \geq \cdots \geq \lambda_d(\mu)$ for its Lyapunov exponents. The latter ones are provided by Oseledeč's theorem which will be recalled in details later.

Definition 1.0.11. *We define the* ***Lyapunov dimension*** *$D(\mu)$ of an ergodic measure μ in the following way. Let*

$$k = \max\{i : 0 < h_\mu + \lambda_1(\mu) + \cdots + \lambda_i(\mu)\}$$

. 1. *If $k < d$ then*

$$D(\mu) := k + \frac{h_\mu + \lambda_1(\mu) + \cdots + \lambda_k(\mu)}{-\lambda_{k+1}(\mu)}.$$

2. *If $k = d$, thus $h_\mu + \lambda_1(\mu) + \cdots + \lambda_d(\mu) > 0$, then*

$$D(\mu) := d \cdot \frac{h_\mu}{-(\lambda_1(\mu) + \cdots + \lambda_d(\mu))}.$$

Note that this dimension is the extension of the similarity dimension for self-affine cases. Figure 1.1 might give a better understanding of the Lyapunov dimension.

Let μ be an ergodic invariant measure. We define the **t-energy** of μ for $t \geq 0$ by

$$E_\mu(t) := \lim_{n \to \infty} \frac{1}{n} \sum_{|\mathbf{i}|=n} \mu(\mathbf{i}) \log \phi^t(A_\mathbf{i}).$$

Theorem 1.0.1 (Käenmäki, [8])**.** *For $t \geq 0$ we have*

$$P(\mathcal{T}, t) = \sup_\mu \{h_\mu + E_\mu(t)\},$$

where h_μ denotes the entropy of μ, and $\mathcal{T}$ is the same as in Definition 1.0.8.

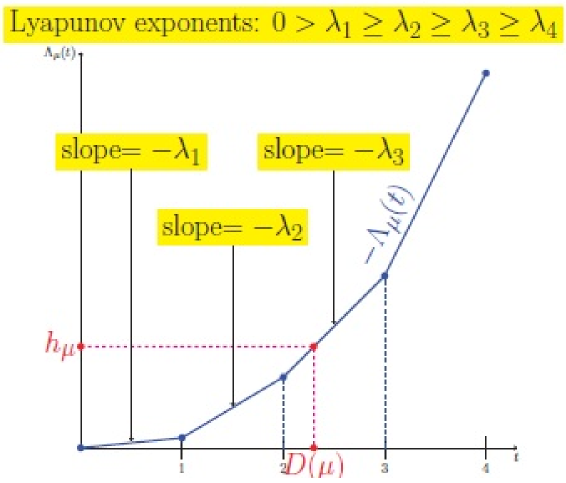

Figure 1.1: This figure from [14] shows us the meaning of the Lyapunov dimension.

Definition 1.0.12. *We call an ergodic measure μ_K on Σ* ***Käenmäki measure****, if it satisfies*

$$0 = P(\mathcal{T}, \dim_A \Lambda) = h_{\mu_K} + E_{\mu_K}(\dim_A \Lambda).$$

Applying Theorem 1.0.1 it follows that μ_K exists, and it is known from [8] that $D(\mu_K) = \dim_A \Lambda$. In most cases we will use a Bernoulli measure μ defined by a probability vector $\mathbf{p} = (p_1, \ldots, p_m)$ on Σ as an ergodic invariant measure, but at some parts of this thesis we will use the Käenmäki measure instead.

Chapter 2

Graph directed IFS

In this chapter we will follow Falconer's book [5] to introduce the graph directed iterated function systems (or GDIFS for short), which will be really important in the next chapter. Most of our results about the broken IFS will depend on the theorems stated here.

To phrase the definition of graph directed iterated function systems we will need a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. We label the vertices of this graph with the numbers $\{1, 2, ..., q\}$, where $|\mathcal{V}| = q$. This $\mathcal{G}$ graph is not assumed to be simple, it might have multiple nodes between the same vertices, or even loops. Denote with $\mathcal{E}_{i,j}$ the set of directed nodes from vertex i to vertex j, and write $\mathcal{E}_{i,j}^k$ for the set of k length directed paths between i and j. We assume that some sort of transversality applies for this graph, which will ensures that there exists at least one directed path between any two vertices. Formally we assume that there exists a $p_0 > 0$ integer, for which the following holds

$$\forall i, j \in \mathcal{V},\ \exists p \in \mathbb{N}, 1 \leq p \leq p_0 : \mathcal{E}_{i,j}^p \neq \emptyset$$

For all $e \in \mathcal{E}$ nodes let $F_e : \mathbb{R}^n \to \mathbb{R}^n$ be a contraction with contraction rate $0 < r_e < 1$. In this case there exist non-empty compact sets $E_1, ..., E_q$ labeled by the elements of $\mathcal{V}$, for which

$$E_i = \bigcup_{j=1}^{q} \bigcup_{e \in \mathcal{E}_{i,j}} F_e(E_j) \tag{2.1}$$

This formula shows that one can think about graph directed iterated function systems as the generalization of the self-similar iterated function systems.

Definition 2.0.1. *The **graph directed iterated function system (GDIFS)** corresponding to the directed graph $\mathcal{G} = \mathcal{V}, \mathcal{E}$ is $\{F_e.e \in \mathcal{E}\}$. We call the sets $\{E_1, ..., E_q\}$ **Graph-directed sets**.*

By iterating (2.1) we get the following equation:

$$E_i = \bigcup_{j=1}^{q} \bigcup_{(e_1,...e_k) \in \mathcal{E}_{i,j}^k} F_{e_1} \circ ... \circ F_{e_k}(E_j) \tag{2.2}$$

From now on we assume that the strong separation property holds for our GDIFS, that is the unions in (2.2) are disjoint. This constriction can be made less strict by expecting the open set condition to hold instead of the SSP.

Our goal now is to achieve a formula for the dimension of such GDIFS-s. To this end we define a $q \times q$ matrix with the following entries

$$A_{i,j}^{(s)} = \sum_{e \in \mathcal{E}_{i,j}} r_e^s. \tag{2.3}$$

The parameter s in the definition of the matrix $A^{(s)}$ will coincide with the Hausdorff-dimension of the attractor, if the matrix satisfies a certain condition. This property is stated in the following theorem.

Theorem 2.0.1 (Corollary 3.5 of [5]). *Let $E_1, ..., E_q$ be graph-directed sets for the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Then there exists an s for which*

$$\dim_H E_i = \underline{\dim}_B E_i = \overline{\dim}_B E_i, \ 0 < \mathcal{H}^s(E_i) < \infty$$

holds for all $i \in \mathcal{V}$. If $A^{(s)}$ is the matrix assigned to this GDIFS, then s is the only number for which $\rho(A^{(s)}) = 1$, where ρ denotes the spectral radius.

Later both this theorem and its proof will be important. The classical results stated by the following two theorems give us tools that will help us prove Theorem 2.0.1.

Theorem 2.0.2 ([5], Theorem 3.1)**.** *Let $E \subset \mathbb{R}^n$ be a non-empty compact set, and fix $a > 0, r_0 > 0$. Assume that for all U with $E \cap U \neq \emptyset$, $|U| < r_0$ there exists a $g : E \cap U \to E$ such that*

$$a|U|^{-1}|x - y| \leq |g(x) - g(y)|$$

holds for all $x, y \in E \cap U$. In this case if $s = \dim_H E$, then $\mathcal{H}^s(E) \geq a^s > 0$ and $\underline{\dim}_B E = \overline{\dim}_B E = s$.

Theorem 2.0.3 ([5], Theorem 3.2)**.** *Let $E \subset \mathbb{R}^n$ be a non-empty compact set, and fix two constants $a > 0, r_0 > 0$. Assume that for all closed balls centered in E with radius $r < r_0$ there exists a $g : E \to E \cap B$ mapping such that*

$$ar|x - y| \leq |g(x) - g(y)|$$

holds for all $x, y \in E$. In this case if $s = \dim_H E$ then $\mathcal{H}^s(E) \leq 4^s a^{-s} < \infty$ and $\underline{\dim}_B E = \overline{\dim}_B E = s$.

Proof. (Proof of theorem 2.0.1) The transversality condition we assumed about the GDIFS provides that for all $(i, j) : 1 \leq i, j \leq q$ pairs there exists a similarity

$$F_{e_1} \circ ... \circ F_{e_p} : E_i \to E_j, \tag{2.4}$$

where $(e_1, ..., e_p) \in \mathcal{E}^p_{i,j}$, and $p \leq p_0$. It follows that $\dim_H E_i \geq \dim_H E_j$ holds for all (i, j) pairs. Thus there exists an s such that $s = \dim_H E_i$ for all i.

First we need to show the $0 < \mathcal{H}^s(E_i) < \infty$ bounds. It suffices to check the assumptions of Theorem 2.0.2 and Theorem 2.0.3. Let $r_{min} = \min_{e \in \mathcal{E}_{i,j}} r_e$ denote the minimum of the contraction rates of the functions between E_j and E_i. Using (2.2) we observe that for a given $x \in E_i$ and $r \leq |E_i|$ there exists a j integer and an $(e_1, ..., e_k) \in \mathcal{E}_{i,j}^k$ path of length k such that $x \in F_{e_1} \circ ... \circ F_{e_k}(E_j)$. We know that for all $e \in \mathcal{E} : r_e < 1$ thus we can find a k for which $rr_{min}|E_i|^{-1} \leq r_{e_1} r_{e_2} \cdot ... \cdot r_{e_k} \leq r|E_i|^{-1}$. Since (2.4) provides an $(e_{k+1}, ..., e_{k+p}) \in \mathcal{E}_{j,j}^p$, $p \leq p_0$ we obtain $x \in F_{e_1} \circ ... \circ F_{e_{k+p}}(E_i)$. The contraction ratio of $F_{e_1} \circ ... \circ F_{e_{k+p}} : E_i \to E_i \cap B(x, r)$ is at least $rr_{min}^{p_0+1}|E_i|^{-1}$, because $r|E_i|^{-1} \geq r_{e_1} \cdot ... \cdot r_{e_k} r_{e_{k+1}} \cdot ... \cdot r_{e_{k+p}} > rr_{min}|E_i|^{-1} r_{min}^{p_0}$. Now we can apply Theorem 2.0.3 with $g = F_{e_1} \circ ... \circ F_{e_{k+p}}$ and $a = r_{min}^{p_0+1}|E_i|^{-1}$ to obtain $s = \dim_H E_i = \underline{\dim}_B E_i = \overline{\dim}_B E_i$ and $\mathcal{H}^s(E_i) < \infty$. Since $\forall i : \dim_H E_i = s$ we conclude that $\forall i : \mathcal{H}^s(E_i) < \infty$.

Now we proceed to the $0 < \mathcal{H}^s(E_i)$ bounds. First fix an index i and let $d = \min \varrho (F_e(E_j), F_{e'}(E_{j'}))$ where the minimum goes over all the different $e \in \mathcal{E}_{i,j}, e' \in \mathcal{E}_{i,j'}$ nodes, and ϱ is the natural metric between the subsets of $\mathbb{R}^n$. Then

$$\varrho \left(F_{e_1} \circ ... \circ F_{e_k}(E_j), F_{e'_1} \circ ... \circ F_{e'_k}(E_{j'}) \geq dr_{e_1}...r_{e_{k-1}}\right) \tag{2.5}$$

holds for different $(e_1, ...e_k) \in \mathcal{E}_{i,j}^k$, $(e'_1, ..., e'_k) \in \mathcal{E}_{i,j'}^k$. Let U be a set intersecting E with $|U| < d$. By (2.2) for an arbitrary $x \in E_i \cap U$ there exist j, k and $(e_1, ..., e_k) \in \mathcal{E}_{i,j}^k$ such that $x \in F_{e_1} \circ ... \circ F_{e_k}(E_j)$ and $dr_{e_1}...r_{e_k} \leq |U| \leq dr_{e_1}...r_{e_{k-1}}$. It follows from (2.5) that $E_i \cap U \subset F_{e_1} \circ ... \circ F_{e_k}(E_j)$, hence $(F_{e_1} \circ ... \circ F_{e_k})^{-1} : E_i \cap U \to E_j$. According to (2.4) there exists $F_{e_{k+1}} \circ ... \circ F_{e_{k+p}} : E_j \to E_i$ where $p \leq p_0$, so we can construct a similarity between $E_i \cap U$ and E_i in the following way

$$\left(F_{e_{k+1}} \circ ... \circ F_{e_{k+p}}\right) \left(F_{e_1} \circ ... \circ F_{e_k}\right)^{-1} : E_i \cap U \to E_i.$$

The rate of this similarity is $(r_{e_{k+1}}...r_{e_{k+p}})(r_{e_1}...r_{e_k})^{-1} \geq dr_{min}^{p_0}|U|^{-1}$. Thus we can apply Theorem 2.0.2 with $g = \left(F_{e_{k+1}} \circ ... \circ F_{e_{k+p}}\right) \left(F_{e_1} \circ ... \circ F_{e_k}\right)^{-1}$ and $a = dr_{min}^{p_0}$ to conclude $\mathcal{H}^s(E_i) > 0$ for all i. Since $\forall i : \dim_H E_i = s$ we conclude that $\forall i : \mathcal{H}^s(E_i) > 0$.

The only thing left to be proved is that $s = \dim_H E_i$ if and only if $\rho(A^{(s)}) = 1$. We assumed that the union in (2.1) is disjoint thus using the additivity and the scaling property of the Hausdorff measure we obtain

$$\begin{aligned}\mathcal{H}^s(E_i) &= \sum_{j=1}^{q} \sum_{e \in \mathcal{E}_{i,j}} \mathcal{H}^s(F_e(E_j)) \\ &= \sum_{j=1}^{q} \sum_{e \in \mathcal{E}_{i,j}} r_e^s \mathcal{H}^s(E_j)\end{aligned}$$

We can write it in matrix form:

$$\begin{bmatrix} \mathcal{H}^s(E_1) \\ \vdots \\ \mathcal{H}^s(E_q) \end{bmatrix} = A^{(s)} \begin{bmatrix} \mathcal{H}^s(E_1) \\ \vdots \\ \mathcal{H}^s(E_q) \end{bmatrix},$$

where $A^{(s)}$ is the matrix assigned to the GDIFS defined in (2.3). According to the Perron-Frobenius Theorem any matrix with non-negative entries has a non-negative eigenvector which is unique to within a scalar multiple and which corresponds to the biggest eigenvalue. In our case $A^{(s)}$ has non-negative entries, and if we choose $s = \dim_H E_i$ then $(\mathcal{H}^s(E_1), ..., \mathcal{H}^s(E_q))^T$ is the non-negative eigenvector of $A^{(s)}$ corresponding to the eigenvalue 1. This formula uniquely determines s since $\rho(A^{(s)})$ is strictly decreasing in s.

□

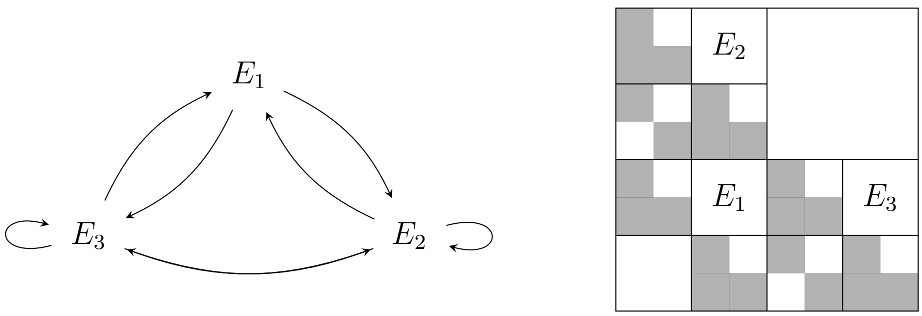

Figure 2.1: The graph of the example IFS, and the second cylinders.

Example 2.0.1. *Consider a graph directed IFS defined on* $[0,1]^2$. *The graph directed sets are three of the quarters of the unit square, and all the functions of this system has contraction ratio* $1/2$. *The second cylinders, and the graph of the system are presented on Figure 2.1. The matrix of the system is*

$$\begin{bmatrix} 0 & \left(\frac{1}{2}\right)^s & \left(\frac{1}{2}\right)^s \\ \left(\frac{1}{2}\right)^s & \left(\frac{1}{2}\right)^s & \left(\frac{1}{2}\right)^s \\ \left(\frac{1}{2}\right)^s & \left(\frac{1}{2}\right)^s & \left(\frac{1}{2}\right)^s \end{bmatrix}.$$

We can calculate its dimension s *using Theorem 2.0.1 :*

$$s = -\frac{\log(\sqrt{3}-1)}{\log 2}.$$

Chapter 3

Broken systems

In this chapter we will investigate the dimension theory of a certain family of non-conformal IFS. We will call this family Broken Iterated Function Systems. To emphasize the reason why we are interested in this family, and to show where did their name come from let us first look at the smooth IFS in the following example. Throughout this chapter we will assume that the SSP hold for the mentioned systems.

Example 3.0.1. *Let* $\mathcal{F} = \{f_i(x) = \lambda_i x + t_i\}_{i=1}^{3}$ *be defined by the following similarities on* $[0,1]$*:*

$$f_1(x) = \frac{1}{4}x,\ f_2(x) = \frac{1}{4}x + \frac{3}{8},\ f_3(x) = \frac{1}{4}x + \frac{3}{4}$$

Figure 3.1 illustrates the functions of $\mathcal{F}$*. On this figure the vertical axis represents the image of the functions, and it is easy to see that this IFS satisfies the SSP. The identical mapping is also on the figure, marked with a dashed line, to make the fixed points of the functions* f_i *visible.*

The dimension theory of such self-similar systems is well-known. The cylinders are strongly separated, hence the Hausdorff dimension of the attractor equals to the similarity dimension. It means that the following equality gives the Hausdorff dimension of $\mathcal{F}$*:*

$$\left(\frac{1}{4}\right)^s + \left(\frac{1}{4}\right)^s + \left(\frac{1}{4}\right)^s = 3 * \left(\frac{1}{4}\right)^s = 1 \Rightarrow s = \frac{\log\frac{1}{3}}{\log\frac{1}{4}} = \frac{\log 3}{\log 4}$$

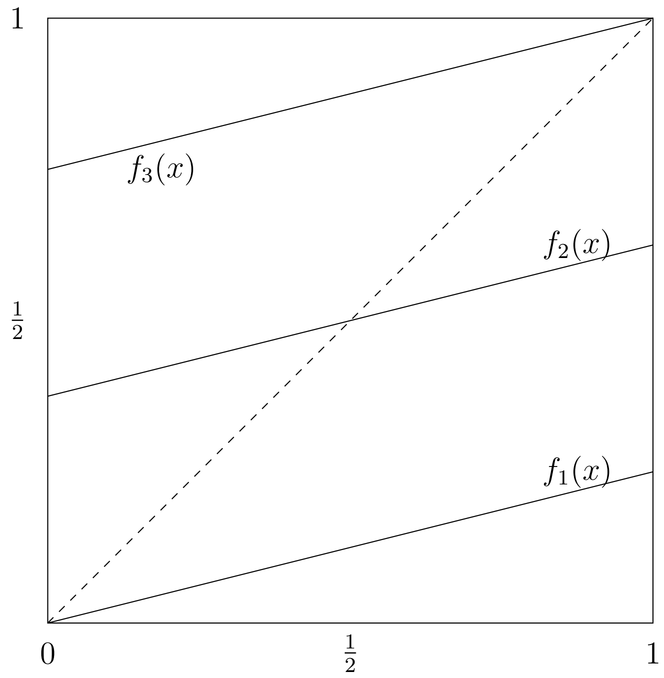

Figure 3.1: The IFS of the example consists of conformal functions.

For $\mathcal{F} = \{f_i(x) = \lambda_i x + t_i\}_{i=1}^m$ with $\forall i : 0 < \lambda_i < 1,\ t_i \in [0,1)$ the dimension can be calculated in a similar fashion. Suppose that the SSP holds and write Λ for the attractor of $\mathcal{F}$, then

$$\dim_H \Lambda = s, \text{ where } \sum_{i=1}^{m} \lambda_i^s = 1.$$

Now we can define the family of our interest. Let $\mathcal{F}^B$ be a **Broken iterated function system**, and define it as the set of the following functions on $[0,1]$:

$$f_1(x) = \lambda_1 x, \qquad f_3(x) = \lambda_3 x + (1-\lambda_3) \qquad f_2(x) = \begin{cases} \lambda_{2,l}x + \frac{1}{2}(1-\lambda_{2,l}), \text{ if } x \in \left[0, \frac{1}{2}\right] \\ \lambda_{2,r}x + \frac{1}{2}(1-\lambda_{2,r}), \text{ if } x \in \left[\frac{1}{2}, 1\right] \end{cases}$$

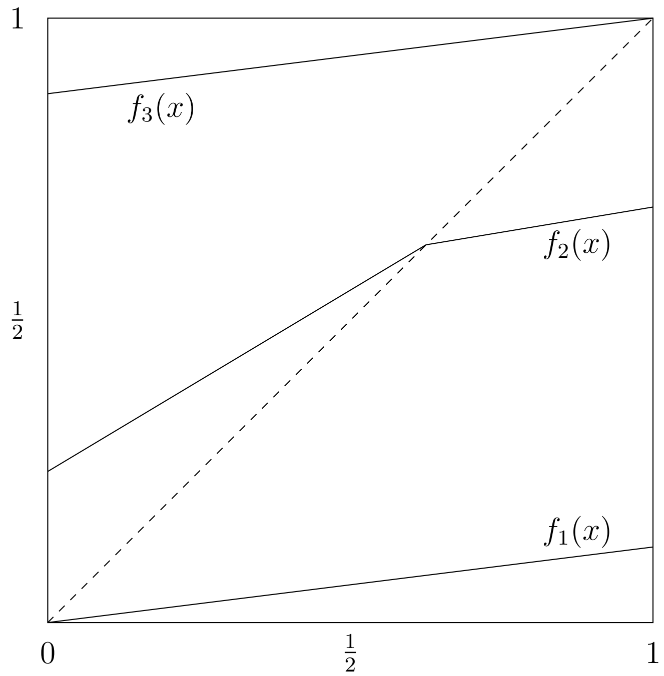

Figure 3.2: The functions of a broken IFS with $\lambda_{2,l} > \lambda_{2,r}$.

The functions will be similar to the ones shown on figure 3.2 given that $\lambda_{2,l} > \lambda_{2,r}$. Opposed to Example 3.0.1 in this case f_2 is not conformal, since it has different ratios on the different sides of its fixed point. Because of this we need a new formula to calculate the Hausdorff dimension of the attractor.

3.1 Constructing a graph

We will calculate the dimension of broken systems using the method introduced in the previous chapter. Observe that $\mathcal{F}^B$ is a GDIFS with the directed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, where

$$\mathcal{V} = \{1, 2, 3, 4\}, \mathcal{E} = \{e_{1,1}, e_{1,2}, e_{1,3}, e_{1,4}, e_{2,1}, e_{2,2}, e_{3,3}, e_{3,4}, e_{4,1}, e_{4,2}, e_{4,3}, e_{4,4}\}$$

. First we need to split f_2 into 2 functions: $f_{2,l}, f_{2,r}$. Now the first cylinders of the system $\mathcal{F}'^B = \{f_1, f_{2,l}, f_{2,r}, f_3\}$ will be the graph directed sets $E_1, E_{2,l}, E_{2,r}, E_3$.

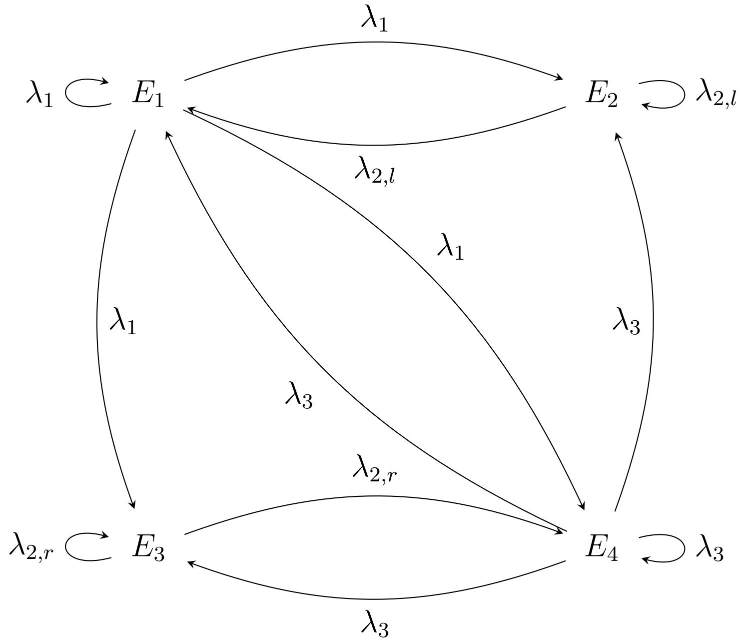

Figure 3.3: The graph $\mathcal{G}$ of $\mathcal{F'}^B$ with the contraction ratios of the corresponding functions on the edges.

We found the graph that directs the system $\mathcal{F}^B$, next we proceed to use the Theorem from the previous chapter that can help us calculate its dimension. Write $A^{(s)}$ for the matrix of the system where

$$A^{(s)} = \begin{bmatrix} \lambda_1^s & \lambda_1^s & \lambda_1^s & \lambda_1^s \\ \lambda_{2,l}^s & \lambda_{2,l}^s & 0 & 0 \\ 0 & 0 & \lambda_{2,r}^s & \lambda_{2,r}^s \\ \lambda_3^s & \lambda_3^s & \lambda_3^s & \lambda_3^s \end{bmatrix}$$

According to Theorem 1.0.1 we need to find an s such that $\rho(A^{(s)}) = 1$. The Perron-Frobenius Theorem and the proof of Theorem 1.0.1. implies that it is enough to find the s for which 1 is an eigenvalue of $A^{(s)}$. In other words we need an s for which $x = 1$ is a solution of the characteristic equation

$\det(A^{(s)} - xI) = 0$, where I is the 4×4 identity matrix.

$$0 = \det \begin{bmatrix} \lambda_1^s - 1 & \lambda_1^s & \lambda_1^s & \lambda_1^s \\ \lambda_{2,l}^s & \lambda_{2,l}^s - 1 & 0 & 0 \\ 0 & 0 & \lambda_{2,r}^s - 1 & \lambda_{2,r}^s \\ \lambda_3^s & \lambda_3^s & \lambda_3^s & \lambda_3^s - 1 \end{bmatrix}$$
$$= -\lambda_{2,l}^s \det \begin{bmatrix} \lambda_1^s & \lambda_1^s & \lambda_1^s \\ 0 & \lambda_{2,r}^s - 1 & \lambda_{2,r}^s \\ \lambda_3^s & \lambda_3^s & \lambda_3^s - 1 \end{bmatrix} + (\lambda_{2,l}^s - 1) \det \begin{bmatrix} \lambda_1^s - 1 & \lambda_1^s & \lambda_1^s \\ 0 & \lambda_{2,r}^s - 1 & \lambda_{2,r}^s \\ \lambda_3^s & \lambda_3^s & \lambda_3^s - 1 \end{bmatrix}$$
$$= -\lambda_{2,l}^s(-\lambda_1^s\lambda_{2,r}^s + \lambda_1^s) + (\lambda_{2,l}^s - 1)(-\lambda_1^s\lambda_{2,r}^s + \lambda_1^s + \lambda_{2,r}^s + \lambda_3^s - 1)$$
$$= 1 - \lambda_1^s - \lambda_{2,l}^s - \lambda_{2,r}^s - \lambda_3^s + \lambda_1^s\lambda_{2,r}^s + \lambda_{2,l}^s\lambda_{2,r}^s + \lambda_{2,l}^s\lambda_3^s$$

Thus the Hausdorff dimension of $\mathcal{F}^B$ is the unique number s that satisfies the equation

$$1 - \lambda_1^s - \lambda_{2,l}^s - \lambda_{2,r}^s - \lambda_3^s + \lambda_1^s\lambda_{2,r}^s + \lambda_{2,l}^s\lambda_{2,r}^s + \lambda_{2,l}^s\lambda_3^s = 0.$$

It worth to check if this formula is consequent with the one used for smooth families like Example 3.0.1. To this end let $\lambda_{2,l} = \epsilon\lambda_{2,r}$ and check our formula as $\epsilon \to 1$.

$$\lim_{\epsilon\to 1} 1 - \lambda_1^s - \epsilon^s\lambda_{2,r}^s - \lambda_{2,r}^s - \lambda_3^s + \lambda_1^s\lambda_{2,r}^s + \epsilon^s\lambda_{2,r}^s\lambda_{2,r}^s + \epsilon^s\lambda_{2,r}^s\lambda_3^s$$
$$= 1 - \lambda_1^s - \lambda_{2,r}^s - \lambda_{2,r}^s - \lambda_3^s + \lambda_1^s\lambda_{2,r}^s + \lambda_{2,r}^s\lambda_{2,r}^s + \lambda_{2,r}^s\lambda_3^s$$
$$= 1 - \lambda_1^s - \lambda_{2,r}^s - \lambda_3^s - \lambda_{2,r}^s(1 - \lambda_1^s - \lambda_{2,r}^s - \lambda_3^s)$$

Writing $a(s) = 1 - \lambda_1^s - \lambda_{2,r}^s - \lambda_3^s$ we get $a(s) = \lambda_{2,r}^s a(s)$. Since $s > 0$ and $\lambda_{2,r} < 1$ the only solution must satisfy $a(s) = 0$. A system $\{g_1, g_2, g_3\}$ with ratios $\lambda_1, \lambda_{2,r}, \lambda_3$ have Hausdorff dimension s exactly if $a(s) = 0$ provided that the SSP holds. Thus our formula for Broken IFS works for smooth systems too.

3.2 General formula

We want to give a similar formula for Broken IFS of more then 3 functions. Each system of this family consists of the following type of functions: One with fixed point 0, another with fixed point 1, and several functions with different ratios on the different sides of their fixed points. We call the later kind broken functions. To make the notation easier let us consider each broken function as two separate functions. Because of this the number of functions in each Broken IFS is even. We will give a formula for the dimension of Broken IFS consisting of $2n$ functions. Write $A_{2n}^{(s)}$ for the matrix of the Broken IFS of $2n$ functions. We know already that the matrix in the simplest case is

$$A_4^{(s)} = \begin{bmatrix} \lambda_1^s & \lambda_1^s & \lambda_1^s & \lambda_1^s \\ \lambda_{2,l}^s & \lambda_{2,l}^s & 0 & 0 \\ 0 & 0 & \lambda_{2,r}^s & \lambda_{2,r}^s \\ \lambda_3^s & \lambda_3^s & \lambda_3^s & \lambda_3^s \end{bmatrix},$$

and we can construct the matrix of the bigger systems in a similar way. We named the first cylinders $E_1, ..., E_{2n}$, they are the graph directed sets. The first and the last sets contract all the E_i sets with f_1 and f_{2n}. By definition, a certain graph directed set E_k with $1 < k < 2n$ will only contract the sets with indices $i \leq k$ given that k is an odd number. Similarly it will only be affected by E_i with $i \geq k$ if k is even. From this the form of the matrices follows:

$$A_6^{(s)} = \begin{bmatrix} \lambda_1^s & \lambda_1^s & \lambda_1^s & \lambda_1^s & \lambda_1^s & \lambda_1^s \\ \lambda_2^s & \lambda_2^s & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3^s & \lambda_3^s & \lambda_3^s & \lambda_3^s \\ \lambda_4^s & \lambda_4^s & \lambda_4^s & \lambda_4^s & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5^s & \lambda_5^s \\ \lambda_6^s & \lambda_6^s & \lambda_6^s & \lambda_6^s & \lambda_6^s & \lambda_6^s \end{bmatrix},$$

$$A_{2n}^{(s)} = \begin{bmatrix} \lambda_1^s & \lambda_1^s & \lambda_1^s & \lambda_1^s & \dots & \lambda_1^s & \lambda_1^s \\ \lambda_2^s & \lambda_2^s & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \lambda_{2n-1}^s & \lambda_{2n-1}^s \\ \lambda_{2n}^s & \lambda_{2n}^s & \lambda_{2n}^s & \lambda_{2n}^s & \dots & \lambda_{2n}^s & \lambda_{2n}^s \end{bmatrix}.$$

Just like previously, we want an s for which $\det(A_{2n}^{(s)} - I_{2n}) = 0$, where I_{2n} is the $2n \times 2n$ identity matrix. Write Form_1^{2i} for the formula of the dimension of the system determined by $A_{2i}^{(s)}$, thus we already know

$$\mathrm{Form}_1^4 = 1 - \lambda_1^s - \lambda_2^s - \lambda_3^s - \lambda_4^s + \lambda_1^s\lambda_3^s + \lambda_2^s\lambda_3^s + \lambda_2^s\lambda_4^s.$$

It is enough to find Form_1^{2n} since then $\mathrm{Form}_1^{2n} = 0$ will give us the dimension. Expanding the determinant $\det(A_{2n}^{(s)} - I_{2n})$ by the $(2n-1)$-th row yields that

$$\mathrm{Form}_1^{2n} = \mathrm{Form}_1^{2(n-1)}(1 - \lambda_{2n}^s - \lambda_{2n-1}^s) - \mathrm{Rem}_{2n},$$

since the determinant of the 2×2 lower left block multiplied by the determinant of the $2(n-1) \times 2(n-1)$ upper right block equals to the right term of the sum, and let Rem_{2n} denote the terms left to be calculated. By the construction of the matrices, and in particular the last two columns, Rem_{2n} must consist of those terms of $\lambda_{2n}^s \cdot \mathrm{Form}_1^{2(n-1)}$ which contains at least one λ_i^s multiplier with an odd i. Thus we obtained the general formula for the dimension by determinant expansion.

We can get this formula with a more elegant way using the patterns these matrices follow. We only show it for $A_6^{(s)}$ since the same argument works for any $A_{2n}^{(s)}$ by induction. Recall that we already have

$$\mathrm{Form}_1^6 = (1 - \lambda_6^s - \lambda_5^s)\mathrm{Form}_1^4 - \mathrm{Rem}_6, \tag{3.1}$$

and we want to see which terms Rem_6 consists of. We calculate the determinant by partitioning $A_6^{(s)} - I_6$ into different blocks. The distinct subdivisions will show us which terms should be excluded from the final formula. Partition $A_6^{(s)} - I_6$ into blocks in the following ways

$$a)\left[\begin{array}{cccc|cc}
\lambda_1^s-1 & \lambda_1^s & \lambda_1^s & \lambda_1^s & \lambda_1^s & \lambda_1^s\\
\lambda_2^s & \lambda_2^s-1 & 0 & 0 & 0 & 0\\
0 & 0 & \lambda_3^s-1 & \lambda_3^s & \lambda_3^s & \lambda_3^s\\
\lambda_4^s & \lambda_4^s & \lambda_4^s & \lambda_4^s-1 & 0 & 0\\
\hline
0 & 0 & 0 & 0 & \lambda_5^s-1 & \lambda_5^s\\
\lambda_6^s & \lambda_6^s & \lambda_6^s & \lambda_6^s & \lambda_6^s & \lambda_6^s-1
\end{array}\right],$$

$$b)\left[\begin{array}{cc|cccc}
\lambda_1^s-1 & \lambda_1^s & \lambda_1^s & \lambda_1^s & \lambda_1^s & \lambda_1^s\\
\lambda_2^s & \lambda_2^s-1 & 0 & 0 & 0 & 0\\
\hline
0 & 0 & \lambda_3^s-1 & \lambda_3^s & \lambda_3^s & \lambda_3^s\\
\lambda_4^s & \lambda_4^s & \lambda_4^s & \lambda_4^s-1 & 0 & 0\\
0 & 0 & 0 & 0 & \lambda_5^s-1 & \lambda_5^s\\
\lambda_6^s & \lambda_6^s & \lambda_6^s & \lambda_6^s & \lambda_6^s & \lambda_6^s-1
\end{array}\right].$$

We obtained (3.1) using partition $a)$ and partition $b)$ yields a similar formula, namely

$$\text{Form}_1^6 = (1-\lambda_2^s-\lambda_1^s)\text{Form}_3^6 - \text{Rem}'_{2n}, \qquad (3.2)$$

where Rem'_6 denotes the new reminders. Since Form_3^6 do not contain $\lambda_6^s\lambda_3^s$ we conclude that it must be in Rem_6 with any terms of the form $\lambda_6^s\lambda_3^s\lambda_i^s$.

$$c)\left[\begin{array}{cc|cc|cc}
\lambda_1^s-1 & \lambda_1^s & \lambda_1^s & \lambda_1^s & \lambda_1^s & \lambda_1^s\\
\lambda_2^s & \lambda_2^s-1 & 0 & 0 & 0 & 0\\
\hline
0 & 0 & \lambda_3^s-1 & \lambda_3^s & \lambda_3^s & \lambda_3^s\\
\lambda_4^s & \lambda_4^s & \lambda_4^s & \lambda_4^s-1 & 0 & 0\\
\hline
0 & 0 & 0 & 0 & \lambda_5^s-1 & \lambda_5^s\\
\lambda_6^s & \lambda_6^s & \lambda_6^s & \lambda_6^s & \lambda_6^s & \lambda_6^s-1
\end{array}\right]$$

If we use partition $c)$ and calculate first the determinant of the middle block, it follows similarly that Rem_6 includes all the terms containing $\lambda_6^s\lambda_1^s$. With this reasoning we obtained the same formula for Form_1^6, since Rem_6 consists of the same elements.

If we write $\lambda^s_{i_1\dots i_n} := \lambda^s_{i_1}\dots\lambda^s_{i_n}$ then the general formula can be written in the following form:

$$\mathrm{Form}_1^{2n} = \mathrm{Form}_1^{2(n-1)}(1-\lambda^s_{2n}-\lambda^s_{2n-1}) - \lambda^s_{2n}\sum_{\mathbf{i}\in V}(-1)^{|\mathbf{i}|+1}\lambda^s_{\mathbf{i}},$$

where V is the set of the words with length at most $(n-1)$ that contains at least one λ^s_i with odd index. Thus we get the Hausdorff dimension of the system which corresponds to $A^{(s)}_{2n}$ if we calculate

$$\mathrm{Form}_1^{2n} = 0.$$

Chapter 4

The Furstenberg measure

Since the notion of the Furstenberg measure can be a bit complicated for the first grasp, we will introduce it by an example. Before that, we need to recall Oseledec's well known multiplicative ergodic theorem [10], and the notation related to this high impact result. Throughout this chapter all the definitions and theorems use the same notation.

4.1 The Oseledec Theorem

Let f be a diffeomorphism of a compact manifold M.

Definition 4.1.1. *We call an $x \in M$ a **regular point** of f, if there exist numbers $\lambda_1(x) > \lambda_2(x) > \cdots > \lambda_m(x)$ and a decomposition*

$$T_xM = E_1(x) \oplus E_2(x) \oplus \cdots \oplus E_m(x) \tag{4.1}$$

such that for all $1 \leq j \leq m$ and all $0 \neq u \in E_j(x)$:

$$E_j(f(x)) = (D_xf)E_j(x), \quad \lim_{n \to +\infty} \frac{1}{n} \log\|(D_xf^n)u\| = \lambda_j(x). \tag{4.2}$$

It is easy to see that the λ_i numbers, and the decomposition of the above definition is unique. To prove it, choose an arbitrary $0 \neq u \in T_xM$. Using (4.1) we can write $u = \sum_{j=1}^{m} u_j$, where $\forall j : u_j \in E_j(x)$.

Let $j := \min\{1 \leq i \leq m : u_i \neq 0\}$, and $k := \max\{1 \leq i \leq m : u_i \neq 0\}$. In this case

$$\lim_{n\to+\infty} \frac{1}{n} \log\|(D_x f^n)u\| = \lambda_j(x), \text{ and}$$

$$\lim_{n\to-\infty} \frac{1}{n} \log\|(D_x f^n)u\| = \lambda_k(x).$$

From these we obtain that

$$\lim_{n\to+\infty} \frac{1}{n} \log\|(D_x f^n)u\| = \lim_{n\to-\infty} \frac{1}{n} \log\|(D_x f^n)u\|$$

holds if and only if $\exists 1 \leq j \leq m : u \in E_j(x)$. This implies the uniqueness of $\{\lambda_i(x)\}_{i=1}^m$ and $\{E_i(x)\}_{i=1}^m$.

We call the numbers $\{\lambda_i(x)\}_{i=1}^m$ **Lyapunov exponents**, and the spaces $\{E_i(x)\}_{i=1}^m$ **eigenspaces** of f at the regular point x. In general, regular points form a set of first category. On contrary, according to the theorem of Oseledec below, this set has total measure with respect to any invariant measure. To give a heuristic picture, we can say that these points define a set which is really small in means of topology, but large in the ergodic point of view.

Theorem 4.1.1 (Oseledeč, [10])**.** *If M is compact, the set of regular points of a diffeomorphism $f : M \to M$ is a Borel set with total measure.*

We also present here a differently formulated version of this theorem from Walter's book [15]. Let $\mathcal{E}_r$ be the space $\mathbb{R}^r$ equipped with the euclidean norm for some fixed r. Write $\mathbf{M}_r$ for the set of $r \times r$ matrices with elements from $\mathbb{R}$. For $A \in \mathbf{M}_r$ we use the norm $\|A\| = \sup\{\|Au\| : u \in \mathcal{E}_r, \|u\| \leq 1\}$, and we denote the transpose of the matrix A with A^*.

Theorem 4.1.2 (Oseledeč, Theorem 10.2 in [15])**.** *Let τ be an endomorphism of a probability space $(\Omega, \mathcal{A}, \mu)$ and $A(\cdot)$ a measurable map $\Omega \to \mathbf{M}_r$ with* $\log^+ \|A(\cdot)\| \in L_1(\mu)$. *There exists a τ-invariant subset Ω' of Ω with $\mu(\Omega') = 1$ such that for $\omega \in \Omega'$ the following is true:*

(i) Let $P_n(A, \omega) := A(\tau^{n-1}\omega) \cdot A(\tau^{n-2}\omega) \cdot \cdots \cdot A(\omega)$, then

$$\lim_{n\to\infty} (P_n^*(A, \omega) P_n(A, \omega))^{1/2n} =: \Lambda(\omega) \in \mathbf{M}_r \text{ exists.}$$

(ii) Let $\exp \lambda_s(\omega) < \cdots < \exp \lambda_2(\omega) < \exp \lambda_1(\omega)$ *be the distinct eigenvalues of* $\Lambda(\omega)$ *in increasing order. (Their number* $s = s(\omega)$ *may depend on* $\omega, \lambda_1(\omega)$ *may be* $-\infty$*.) Let* E_k *be the eigenspace corresponding to* $\exp \lambda_k(\omega)$, $m_k(\omega) = \dim E_k(\omega)$ *the multiplicity of* $\exp \lambda_k(\omega)$, *and* $\mathcal{E}_k(\omega) = E_1(\omega) + \cdots + E_k(\omega)$, $\mathcal{E}_0 = \{0\}$.

then we have

$$\lim_{n\to\infty} \frac{1}{n} \log \|P_n(A,\omega)u\| = \lambda_k(\omega),$$

for $u \in \mathcal{E}_k(\omega) \setminus \mathcal{E}_{k-1}(\omega), \quad (k = 1, \ldots s)$.

(iii) The functions $\omega \to m_k(\omega)$ *and* $\omega \to \lambda_k(\omega)$ *are* τ*-invariant.*

(iv) If τ *is ergodic,* $\det(A(\omega)) \equiv 1$, *and* $\limsup \int \frac{1}{n} \log \|P_n(A,\omega)\| d\mu > 0$, *then* λ_s *is negative and* λ_1 *is positive.*

Let $\mathcal{S} = \{S_i(\mathbf{x}) = A_i\mathbf{x} + \mathbf{t}_i\}_{i=1}^m$ be an IFS on $\mathbb{R}^2$, where $\forall i : \|A_i\| < 1$ non-singular and $\mathbf{t}_i \in \mathbb{R}^2$. If μ is a self-similar measure on the symbolic space $\Sigma = \{1, ..., m\}^{\mathbb{N}}$ then we obtain the corresponding Lyapunov exponents using Oseledec's theorem.

$$\lambda_\mu^1 = \lim_{n\to\infty} \frac{1}{n} \log \|A_{i_1} ... A_{i_n}\|$$
$$\lambda_\mu^2 = \lim_{n\to\infty} \frac{1}{n} \log \|(A_{i_1} ... Ai_n)^{-1}\|^{-1}$$

If we write $\alpha_j(A_i)$, $j = 1, 2$ for the singular values of A_i and use the notation $A_{i_1 \ldots i_n} = A_{i_1} ... A_{i_n}$ then we can state a special case of Oseledec's theorem in the following way.

Theorem 4.1.3 (Oseledec)**.** *For any* $\{A_1, ..., A_m\}$ *set of non-singular* 2×2 *matrices with* $\forall i \in \{1, ..., m\} : \|A_i\| < 1$, *and for any* σ*-invariant measure* μ *on* Σ *there exists constants* $\lambda_\mu^1 \geq \lambda_\mu^2$ *such that for* μ *almost every* $\mathbf{i} \in \Sigma$

$$\lambda_\mu^1 = \lim_{n\to\infty} \frac{1}{n} \log \|A_{i_1 \ldots i_n}\| = \lim_{n\to\infty} \frac{1}{n} \log \alpha_1(A_{i_1 \ldots i_n})$$
$$\lambda_\mu^2 = \lim_{n\to\infty} \frac{1}{n} \log \|(A_{i_1} ... Ai_n)^{-1}\|^{-1} = \lim_{n\to\infty} \frac{1}{n} \log \alpha_2(A_{i_1 \ldots i_n})$$

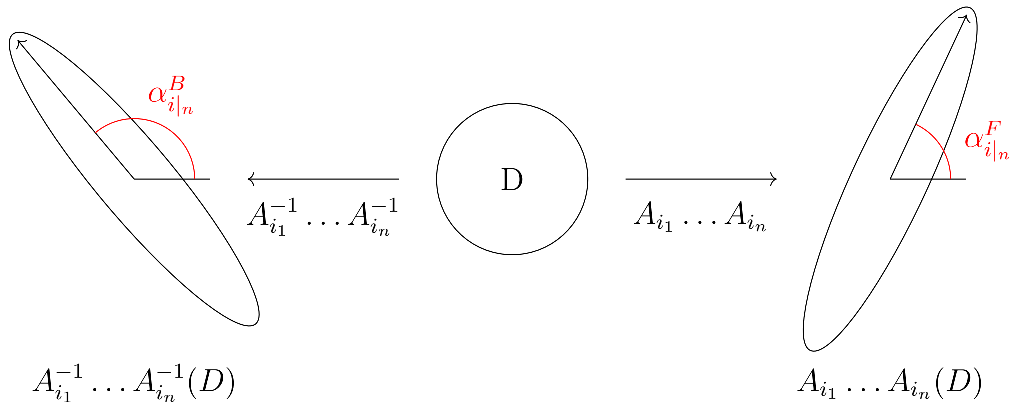

Figure 4.1: A schematic figure of the angles $\alpha^B_{i|n}$ and $\alpha^F_{i|n}$, which are the determiners of the Backward- and Forward Furstenberg measures.

4.2 Introducing the Furstenberg measure

Consider the self-affine IFS $\mathcal{F} = \{f_i(x) = A_i x + t_i\}_{i=1}^m$ on $\mathbb{R}^2$ with all A_i being contractions. Let μ be an ergodic measure on the symbolic space $\Sigma = \{1, \dots, m\}^{\mathbb{N}}$. Assume that we obtained different Lyapunov exponents $0 > \lambda_1(\mu) > \lambda_2(\mu)$ from Oseledec's theorem.

The image of the unit disc D under the composition $A_{i_1} \dots A_{i_n}$ is an ellipse. Write $\alpha^F_{\mathbf{i}|n}$ for the angle of the semi-major axis of this ellipse. We will call the push-forward measure ν_{FF} of μ under the projection Π_{FF} **Forward Furstenberg measure**, where

$$\text{for } \mu\text{-almost all } \mathbf{i} \in \Sigma: \ \Pi_{FF}(\mathbf{i}) := \lim_{n\to\infty} \alpha^F_{\mathbf{i}|n}, \quad \nu_{FF} := (\Pi_{FF})_* \mu. \tag{4.3}$$

In a similar fashion, let us write $\alpha^B_{\mathbf{i}|n}$ for the angle of the semi-major axis of the ellipse $A^{-1}_{i_1} \dots A^{-1}_{i_n}(D)$, and define the **Backward Furstenberg measure** ν_{BF} as follows

$$\text{for } \mu\text{-almost all } \mathbf{i} \in \Sigma: \ \Pi_{BF}(\mathbf{i}) := \lim_{n\to\infty} \alpha^B_{\mathbf{i}|n}, \quad \nu_{BF} := (\Pi_{BF})_* \mu. \tag{4.4}$$

Theorem 4.1.2 provides that these angles are exist in limes, since they are the angles of the eigenspaces E_1 and E_2 which are uniquely determined.

It is really important that we assumed that the Lyapunov exponents $\lambda_1(\mu) > \lambda_2(\mu)$ are not equal. Otherwise we would not get an ellipse with semi-axes of different length after applying $A_{i_1} \dots A_{i_n}$ or $A_{i_1}^{-1} \dots A_{i_n}^{-1}$ to the unit circle. The meaning of the angles $\alpha^F_{\mathbf{i}|_n}$ and $\alpha^B_{\mathbf{i}|_n}$ is shown on Figure 4.1, and a bit later we will see why the measures ν_{FF} and ν_{BF} are useful.

4.3 An IFS of triangular maps

We will follow [9] and define the following self-affine IFS of triangular maps on $\mathbb{R}^2$:

$$\mathcal{F} = \left\{ F_i(\mathbf{x}) = A_i\mathbf{x} + \mathbf{t}_i = \begin{bmatrix} b_i & 0 \\ d_i & a_i \end{bmatrix} \mathbf{x} + \begin{bmatrix} t_{i,1} \\ t_{i,2} \end{bmatrix} \right\}_{i=1}^{m}, \tag{4.5}$$

where $t_{i,1}, t_{i,2} \geq 0$, $d_i \in \mathbb{R}$ and $0 < a_i < b_i < 1$ for all $i \in \{1, ..., m\}$ i.e. direction x-dominates. Without loss of generality we may assume that all functions of this IFS contracts the unit square $R = [0,1]^2$ into itself, that is $f_i(R) \subset R$. Write Λ for the attractor of $\mathcal{F}$, and $\Pi : \Sigma \to \Lambda$ for the natural projection. Let μ be the uniform Bernoulli measure on Σ, and let $\nu := \Pi_*\mu$ be its push forward measure under the natural projection. We use the standard notation for composition of maps and matrix products:

$$F_{i_1 \dots i_n} := F_{i_1} \circ F_{i_2} \circ \dots \circ F_{i_n}$$

$$A_{i_1 \dots i_n} := A_{i_1} \cdot \dots \cdot A_{i_n} := \begin{bmatrix} b_{i_1 \dots i_n} & 0 \\ d_{i_1 \dots i_n} & a_{i_1 \dots i_n} \end{bmatrix},$$

where $i_j \in \{1, ..., m\}$. By immediate calculations we obtain that the terms of $A_{i_1 \dots i_n}$ are $a_{i_1 \dots i_n} = a_{i_1} \cdot \dots \cdot a_{i_n}$, $b_{i_1 \dots i_n} = b_{i_1} \cdot \dots \cdot b_{i_n}$ and

$$d_{i_1 \dots i_n} = \sum_{j=1}^{n} d_{i_j} \cdot \prod_{k<j} a_{i_k} \cdot \prod_{l=j+1}^{n} b_{i_l}, \tag{4.6}$$

where by definition $\prod_{k<1} a_{i_k} := 1$, $\prod_{l=n+1}^{n} b_{i_l} := 1$. The image $R_{i_1 \dots i_n} := f_{i_1 \dots i_n}(R)$ is a parallelogram with two vertical sides, see Figure 4.2.

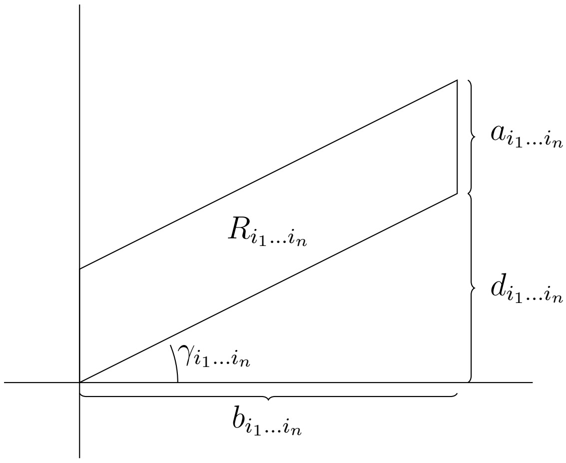

Figure 4.2: The image of $R = [0,1]^2$ under $A_{i_1 \dots i_n}$ for a certain $\mathbf{i} \in \Sigma$.

We will call $b_{i_1 \dots i_n}$ the width, $a_{i_1 \dots i_n}$ the height and $\gamma_{i_1 \dots i_n}$ the angle of the longer side of the parallelogram $R_{i_1 \dots i_n}$, where

$$\tan \gamma_{i_1 \dots i_n} := \frac{d_{i_1 \dots i_n}}{b_{i_1 \dots i_n}}.$$

We assumed that direction x dominates which implies that this parallelogram will get longer and thinner as $n \to \infty$ for any $\mathbf{i} \in \Sigma$. It is easy to see that there is a uniform upper bound for $|\tan \gamma_{i_1 \dots i_n}|$.

Lemma 4.3.1 ([9],Lemma 1.3)**.** *There exists a non-negative $K_0 < \infty$ such that for every $\mathbf{i} = (i_1, \dots i_n) \in \Sigma^*$finite length word*

$$\tan \gamma_{i_1 \dots i_n} = \left| \frac{d_{i_1 \dots i_n}}{b_{i_1 \dots i_n}} \right| \leq K_0.$$

Proof. We assumed $\forall i \in \Sigma : a_i < b_i$ which implies $\max_i \{\frac{a_i}{b_i}\} < 1$. Using (4.6) the statement of the lemma follows.

$$\left| \frac{d_{i_1 \dots i_n}}{b_{i_1 \dots i_n}} \right| \leq \frac{|d_{i_1}|}{b_{i_1}} + \sum_{j=2}^{n} \frac{|d_{i_j}|}{b_{i_j}} \cdot \prod_{j=1}^{k-1} \frac{a_{i_j}}{b_{i_j}} \leq \frac{\max_i \{|d_i|/b_i\}}{1 - \max_i \{a_i/b_i\}} < \infty$$

□

Assume that we have an underlying Bernoulli measure μ generated by the probability vector $(p_1, ..., p_m)$ on Σ. The next lemma helps us calculate the Lyapunov exponents χ_μ^1, χ_μ^2.

Lemma 4.3.2 ([9], Lemma 3.9). *The Lyapunov exponents of the IFS defined above with respect to the Bernoulli measure μ are*

$$\chi_\mu^1 = \sum_{i=1}^{m} p_i \log b_i, \quad \chi_\mu^n = \sum_{i=1}^{m} p_i \log a_i.$$

Proof. First we calculate the singular values. To make the formulas simpler during this proof we use the notation

$$A_{i_1 \ldots i_n} = A = \begin{bmatrix} b & 0 \\ d & a \end{bmatrix}, \quad A_{i_1 \ldots i_n}^{-1} = A^{-1} = \begin{bmatrix} 1/b & 0 \\ -d/(ab) & 1/a \end{bmatrix}.$$

Writing A^T for the transpose of the matrix A we get

$$A^T A = \begin{bmatrix} b^2 + d^2 & ad \\ ad & a^2 \end{bmatrix}.$$

The solutions $\lambda_{1,2}$ of the characteristic equation $det(A^T A - \lambda I) = 0$ are

$$\begin{aligned} \lambda_{1,2} &= \frac{1}{2}\left(a^2 + b^2 + d^2 \pm \sqrt{(a^2 + b^2 + d^2)^2 - 4a^2 b^2}\right) \\ &= \frac{1}{2}\left(a^2 + b^2 + d^2 \pm \sqrt{(-a^2 + b^2 + d^2)^2 + 4a^2 d^2}\right) \\ &= \frac{1}{2}\left(a^2 + b^2 + d^2 \pm |b^2 + d^2 - a^2| \sqrt{1 + \frac{4a^2 b^2}{b^2 + d^2 - a^2}}\right). \end{aligned}$$

Using the expansion $\sqrt{1+x} = 1 + x/2 + O(x^2)$ we obtain

$$\lambda_{1,2} = \frac{1}{2}\left(a^2 + b^2 + d^2 \pm |b^2 + d^2 - a^2| \left(1 + \frac{2a^2 d^2}{(b^2 + d^2 - a^2)^2} + err\right)\right),$$

where err is the error term of the expansion which will tend to zero. We supposed that direction x dominates and it implies that $b^2 > a^2$ thus $|b^2 + d^2 - a^2| = b^2 + d^2 - a^2$. Hence

$$\lambda_1 = b^2 + d^2 + \frac{a^2 d^2}{b^2 + d^2 - a^2}, \text{ and } \lambda_2 = a^2 - \frac{a^2 d^2}{b^2 + d^2 - a^2}$$

Observe that

$$\left|\frac{a^2d^2}{b^2+d^2-a^2}\right| = \left|\frac{a^2}{b^2/d^2+1-a^2/d^2}\right| < \left|\frac{a^2}{b^2/d^2+1-b^2/d^2}\right| = \left|a^2\right|,$$

thus the last term of both of the eigenvalues of A^TA tend to 0 as $n \to \infty$. If n is big enough we have the following formulas for the singular values:

$$\alpha_2(A) = \sqrt{\lambda_2(A)} = a + \epsilon$$

$$\alpha_1(A) = \sqrt{\lambda_1(A)} = b\sqrt{1 + d^2/b^2} = b + \epsilon',$$

where the last equality is provided by lemma 4.3.1, and ϵ, ϵ' goes to zero as $n \to \infty$.

Using Oseledec's theorem we obtain formulas for the Lyapunov exponents:

$$\begin{aligned}
\chi_\mu^1 &= \lim_{n\to\infty} \frac{1}{n} \log \alpha_1(A_{1_1\dots i_n}) = \lim_{n\to\infty} \frac{1}{n} \log b_{i_1\dots i_n} \\
&= \lim_{n\to\infty} \frac{1}{n} \log b_{i_1} \cdot \ldots \cdot b_{i_n} = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log b_{i_j} \\
\chi_\mu^2 &= \lim_{n\to\infty} \frac{1}{n} \log \alpha_2(A_{1_1\dots i_n}) = \lim_{n\to\infty} \frac{1}{n} \log a_{i_1\dots i_n} \\
&= \lim_{n\to\infty} \frac{1}{n} \log a_{i_1} \cdot \ldots \cdot a_{i_n} = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log a_{i_j}.
\end{aligned}$$

To finish the proof we use the strong law of large numbers:

$$\begin{aligned}
\chi_\mu^1 &= \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log b_{i_j} = \sum_{i=1}^{m} p_i \log b_i \\
\chi_\mu^2 &= \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log a_{i_j} = \sum_{i=1}^{m} p_i \log a_i,
\end{aligned}$$

since the underlying measure on Σ was the Bernoulli measure μ generated by the probability vector $\mathbf{p} = (p_1, \ldots, p_m)$.

□

4.4 Connection with Transversality

We will show that the transversality of an IFS can be proved with the help of the Furstenberg measure according to [3]. We want to study the action of $\mathcal{F}$

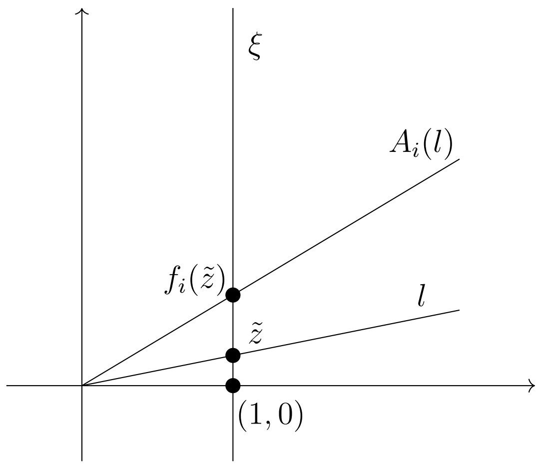

Figure 4.3: This figure shows the connection between the action of A_i on a line l, and the map f_i defined on $\mathbb{R}$.

on the projective line. To do this we need to observe the action of the maps $A_i, i \in \{1, ..., m\}$ on the line $\xi := \{(1, z) \in \mathbb{R}^2 : z \in \mathbb{R})\}$. It is represented on Figure 4.3.

$$A_i \begin{bmatrix} 1 \\ z \end{bmatrix} = \begin{bmatrix} b_i \\ a_i z + d_i \end{bmatrix} = b_i \begin{bmatrix} 1 \\ \frac{a_i}{b_i} z + \frac{d_i}{b_i} \end{bmatrix}$$

Thus after identifying $(1, z) \in \xi$ with $\tilde{z} \in \mathbb{R}$ we can represent it as the following IFS on ξ

$$\left\{ f_i(z) = \frac{a_i}{b_i} z + \frac{d_i}{b_i} \right\}_{i=1}^{m}.$$

We assumed that x-direction dominates thus $\forall i \in \{1, ..., m\} : f_i$ is a contraction, which implies that we can define the natural projection $\Pi_{AF} : \Sigma \to \xi$ in the usual way:

$$\Pi_{AF}(\mathbf{i}) := \lim_{n \to \infty} f_{i_1 \ldots i_n}(0) = \frac{d_{i_1}}{b_{i_1}} + \sum_{j=2}^{\infty} \frac{d_{i_j}}{b_{i_j}} \cdot \prod_{k=1}^{j-1} \frac{a_{i_k}}{b_{i_k}}.$$

Definition 4.4.1. *Consider an ergodic measure μ on Σ. We will call the projection μ_{FF} of μ under Π_{AF} the **Foreward Furstenberg measure** corresponding to μ.*

The functions f_i are in an obvious relation with the angles α_i^F we defined before, since the line determined by $\lim_{n\to\infty} f_{\mathbf{i}|_n}(0)$ and the origin has angle $\lim_{n\to\infty} \alpha_{\mathbf{i}|_n}^F$. Hence this definition is a consequent special case of the one we gave before.

Note that if μ is the self-affine measure of the IFS $\mathcal{F}$ then for any interval $I \subset \xi$ coded with $\mathbf{i}|_n$

$$\mu_{FF}(I) = \mu(\mathbf{i}|_n) = \sum_{i=1}^{m} p_i \mu(\sigma \mathbf{i}|_n) = \sum_{i=1}^{m} p_i \mu_{FF}(f_i^{-1}(I)),$$

for a given probability vector $\mathbf{p} = (p_1, \ldots, p_m)$. It means that μ_{FF} is the stationary measure of the Markov chain defined on ξ which jumps to $f_i(x)$ from a given $x \in \xi$ with transition probability p_i.

We saw that the image of a given point $(1, z)$ of ξ under A_i will not be on ξ anymore, so we need to scale it back to get the corresponding map $\tilde{A}_i$ which maps ξ into itself.

$$\tilde{A}_i(\tilde{z}) := \frac{1}{c_i} \cdot A_i \cdot \begin{bmatrix} 1 \\ z \end{bmatrix},$$

where $\tilde{z} \in \xi$ is $\tilde{z} = (1, z)$. Then by its definition, $\Pi_{AF}(\mathbf{i})$ is the natural projection for $\{\tilde{A}_i\}_{i=1}^m$

$$\Pi_{AF}(\mathbf{i}) = \tilde{A}_{i_1}(\Pi_{AF}(\sigma \mathbf{i})).$$

From now on we will work with smaller family of iterated function systems, namely we will assume that the diagonal elements of the matrices of the system are the same.

Definition 4.4.2. *We say that $\mathcal{F}$ is* ***diagonally homogeneous****, if there exist constants $a, b \in (0,1)$ such that $\forall i \in \{1, ..., m\} : a_i = a,\ b_i = b$.*

Choose two arbitrary words $\mathbf{i}, \mathbf{j} \in \Sigma$ from the symbolic space with $i_1 \neq j_1$, and consider the corresponding n level cylinders $[\mathbf{i}|_n]$ and $[\mathbf{j}|_n]$. We have seen in the previous section that these cylinders are parallelograms with one vertical and one "skew" side. Their angle can be defined as the angle between their non-vertical sides like $|\gamma_{i_1...i_n} - \gamma_{j_1...j_n}|$. The following condition holds if any two cylinders $[\mathbf{i}|_n]$ and $[\mathbf{j}|_n]$ with $i_1 \neq j_1$ either are disjoint or have an angle uniformly bounded away from zero.

Definition 4.4.3. *We say that the diagonally homogeneous IFS $\mathcal{F}$ satisfies the **transversality condition** if there exists a $K > 0$ such that for every n and for every $\mathbf{i}, \mathbf{j} \in \Sigma$ with $i_1 \neq j_1$ we have*

$$(proj_x \circ \Pi)_* \mu([\mathbf{i}|_n] \cap [\mathbf{j}|_n]) < K \cdot a^n, \tag{4.7}$$

where $proj_x$ is the orthogonal projection to the x-axis, and Π is the natural projection.

Now we give a geometric interpretation of this condition with the help of the previously defined $\tilde{A}_i$ maps, which provides a method to check it. Let $\tilde{\mathcal{F}}$ be the following IFS on $\mathbb{R}^3$

$$\tilde{\mathcal{F}} := \{\tilde{F}_i(x, y, z) := (F_i(x, y), \tilde{A}_i(z))\}_{i=1}^m$$

Recall that the IFS $\tilde{\mathcal{F}}$ satisfies the strong separation property (SSP) if its natural projection (now it is $\mathbf{i} \mapsto (\Pi(\mathbf{i}), \Pi_{AF}(\mathbf{i}))$) is a bijection. Equivalently we can say that there exists a non-empty open set V for which

a) $\tilde{F}_i(V) \subset V$ for all $i = 1, ..., m$,

b) $cl(\tilde{F}_i(V)) \cap cl(\tilde{F}_j(V)) = \emptyset$ for all $i \neq j$,

where $cl(V)$ denotes the closure of the set V. We defined the open set condition in a similar fashion in the Introduction chapter. The OSC holds for the IFS $\tilde{\mathcal{F}}$ if there exists a non-empty open set V satisfying

a) $\tilde{F}_i(V) \subset V$ for all $i = 1, ..., m$,

b') $\tilde{F}_i(V) \cap \tilde{F}_j(V) = \emptyset$ for all $i \neq j$.

Lemma 4.4.1 ([3], Lemma 1.2)**.**

1) If $\tilde{\mathcal{F}}$ satisfies the SSP then the transversality condition holds for $\mathcal{F}$.

2) $\tilde{\mathcal{F}}$ *satisfies the SSP , if the transversality condition holds for* $\mathcal{F}$ *and* $\forall i \in \{1, ..., m\} : F_i(R) \subset R$.

Proof. First we prove part 1). We have seen already that the images of R are long and thin parallelograms since direction x dominates. We call the direction of the longer sides of a parallelogram **principal axis**. Since our system is diagonally homogeneous we know that the diagonal terms of a matrix which corresponds to an n-length word $\mathbf{i}|_n$ are b^n and a^n. We want to show that if n is big enough then the parallelograms can only intersect transversally. The usual compactness argument shows that if $\mathbf{i} \mapsto (\Pi(\mathbf{i}), \Pi_{AF}\mathbf{i})$ is a bijection, then for any two words $\mathbf{i}, \mathbf{j} \in \Sigma$ with $i_1 \neq j_1$ there exists an $l > 0$ such that one of the following two consequences hold:

$$dist(\Pi(\mathbf{i}), \Pi(\mathbf{j})) > l,$$
$$dist(\Pi_{AF}(\mathbf{i}), \Pi_{AF}(\mathbf{j})) > l.$$

In the first case we know a bound for the distance of the leftmost vertices. The parallelograms $F_{\mathbf{i}|_n}(R)$, $F_{\mathbf{j}|_n}(R)$ can not reach each other if the horizontal distance of these vertices is more than b^n. Thus by simple geometry we get that the cylinders are separated if $l^2 > b^n$. Using logarithm we get a formula for n:

$$l^2 > b^n \Leftrightarrow 2\log l > n \log b \Leftrightarrow n > \frac{-2\log l}{-\log b}. \tag{4.8}$$

In the second case we have a lower bound for the angle between their principal axes. Similarly to the preceding case we obtain another bound for n, using that we have a trigonometric relation between a^n/b^n and this angle:

$$n > \frac{-2\log l}{-log(a/b)}. \tag{4.9}$$

Thus if $n > N$ where

$$N > \frac{-2\log l}{\min(-\log b, -\log(a/b))},$$

by (4.8) and (4.9) the transversality condition holds, proving statement 1).

Now we prove part 2), and assume that (4.7) holds. We also assume that $\forall i \in \{1,...,m\} : F_i(R) \subset R$ thus there exists $l > 0$ such that $\Lambda \in (l, 1-l)^2$. Fix an interval I for which $\tilde{A}_i(I) \subset I$, and fix a large n to be defined later.

Let $\mathbf{i}, \mathbf{j} \in \Sigma$ be distinct words with $i_1 \neq j_1$. Suppose that the parallelograms $F_{\mathbf{i}|_n}([l/2, 1-l/2]^2)$, $F_{\mathbf{j}|_n}([l/2, 1-l/2]^2)$ intersect each other. These are contained in the larger parallelograms $F_{\mathbf{i}|_n}(R)$ and $F_{\mathbf{i}|_n}(R)$, thus the latter ones share an internal point. This point is at a distance at least $a^n(l/2)$ from the vertical boundary, and $b^n(l/2)$ from the non-vertical boundary of $F_{\mathbf{j}|_n}(R)$. By (4.7) the angle between the principal axes of the parallelograms is at least $K' = K(l/2)$. It means that

$$\tilde{F}_{\mathbf{i}|_n}([l, 1-l]^2 \times I) \cap \tilde{F}_{\mathbf{j}|_n}([l, 1-l]^2 \times I) = \emptyset,$$

if our fixed n is so big that $(b/c)^n|I| < K'/2$. It also holds if $F_{\mathbf{i}|_n}([l/2, 1-l/2]^2)$ and $F_{\mathbf{j}|_n}([l/2, 1-l/2]^2)$ have no intersection. Thus it holds for all pair of words $\mathbf{i}|_n, \mathbf{j}|_n$ with different first digits $i_1 \neq j_1$. Therefore $\tilde{\mathcal{F}}$ satisfies the strong separation property for the set

$$V = \cup_{\mathbf{i}|_{n-1}} \tilde{F}_{\mathbf{i}|_{n-1}}((l, 1-l)^2 \times int(I)),$$

where the union is over all $n-1$ length words and $int(I)$ denotes the interior of I. This completes the proof of statement 2).

□

Once we know what does transversality mean, we can formulate the following theorems. Write $\mathcal{L}eb$ for the Lebesgue measure, and $\nu_x = (\Pi \circ proj_x)_* \mu$ for the projection of the measure μ to the x-axis. Since these results do not have direct connection to the Furstenberg measure we will only mention them. The detailed proofs can be seen in [3].

Theorem 4.4.1 ([3], Theorem 4.1). *Let $\mathcal{F}$ be a self-affine diagonally homogeneous IFS of the form* (4.5)*. Assume that the followings hold:*

(A1) $c > \frac{1}{m}$,
(A2) $b < \frac{1}{m}$,

(A3) $\nu_x \ll \mathcal{L}eb$ *with* L^q *density, for some* $q > 1$ *,*
(A4) $\mathcal{F}$ *satisfies transversality.*

Then

$$\dim_H(\Lambda) = \dim_H(\nu) = 1 + \frac{\log(Nc)}{-\log b}.$$

Theorem 4.4.2 ([3], Theorem 4.2)**.** *Let* $\mathcal{F}$ *be a self-affine diagonally homogeneous IFS of the form* (4.5)*. Assume that the followings hold:*

(B1) $c > \frac{1}{m}$*,*
(B2) $b < c$*,*
(B3) $\nu_x \ll \mathcal{L}eb$ *with* L^q *density, for all* $q > 1$ *,*
(B4) $\mathcal{F}$ *satisfies transversality.*

Then

$$\dim_H(\Lambda) = \dim_H(\nu) = \min\left(2, 1 + \frac{\log(Nc)}{-\log b}\right).$$

4.5 Going backwards

In this section we assume that $0 < b < a < 1$, thus the y-direction dominates. Similarly as we already did with the forward measure, we will define the backward Furstenberg measure for this family. Consider again the line $\xi := \{(1, z) \in \mathbb{R}^2 : z \in \mathbb{R}\}$ and identify $(1, z) \in \xi$ with $\tilde{z} \in \mathbb{R}$. Let $\mathcal{B}$ be the following self-affine IFS on ξ

$$\mathcal{B} := \left\{ g_i(\tilde{z}) := \frac{b}{a}\tilde{z} - \frac{d_i}{b} \right\}_{i=1}^{m}.$$

We define the natural projection $\Pi_{BF} : \Sigma \to \xi$ in the usual way:

$$\Pi_{BF}(\mathbf{i}) := -\frac{d_{i_1}}{b} - \sum_{j=2}^{\infty} \frac{d_{i_k}}{b} \left(\frac{c}{b}\right)^{k-1}.$$

We have that the maps $\hat{A}_i : \xi \to \xi$ represents the action of $\{A_i^{-1}\}_{i=1}^m$ on the projective line, where

$$\hat{A}_i(\tilde{z}) := c \cdot A_i^{-1} \cdot \begin{bmatrix} 1 \\ z \end{bmatrix},$$

if $\tilde{z} \in \xi$ is $\tilde{z} = (1, z)$.

Definition 4.5.1. *Consider an ergodic measure μ on Σ. We will call the projection of μ under Π_{BF} the **Backward Furstenberg measure** corresponding to μ.*

By the same reasoning as we gave for the forward measure, this definition of the backward Furstenberg measure coincides with the schematic one we gave using the angles α_i^B.

We will make a good use of this measure in the proof of the next theorem from [3]. We need to define another separation condition, introduced in [7], before stating the theorem.

Definition 4.5.2. *Let $\Phi = \{\phi_i\}_{i=1}^m$ be a self-similar IFS on the line. Write $\Delta_n(\Phi)$ for the minimum of $\Delta(\mathbf{i},\mathbf{j})$ for distinct $\mathbf{i},\mathbf{j} \in \Sigma_n$, where*

$$\Delta(\mathbf{i},\mathbf{j}) = \begin{cases} \infty, & \phi'_{\mathbf{i}}(0) \neq \phi'_{\mathbf{j}}(0), \\ |\phi_{\mathbf{i}}(0) - \phi_{\mathbf{j}}(0)|, & \phi'_{\mathbf{i}}(0) = \phi'_{\mathbf{j}}(0), \end{cases}$$

*and Σ_n denotes the n-length words. We say that the self-similar IFS Φ satisfies **Hochman's exponential separation condition** if there exists an $\epsilon > 0$ and an index series $n_k \nearrow \infty$ such that*

$$\Delta_{n_k}(\Phi) > \epsilon^{n_k}.$$

Let $\mathcal{H}$ be the horizontal projection of the diagonal system corresponding to $\mathcal{F}$

$$\mathcal{H} := \{h_i(x) := b_i x + t_{i,1}\}_{i=1}^m.$$

Theorem 4.5.1 (Theorem 5.1 of [3])**.** *Let $\mathcal{F}$ be a self-affine diagonally homogeneous IFS of the form (4.5). Assume that the followings hold:*

(C1) $b < \frac{1}{m}$,
(C2) $a > b$,
(C3) $\mathcal{B}$ *satisfies Hochman's exponential separation condition,*
(C4) $\mathcal{H}$ *satisfies Hochman's exponential separation condition,*
(C5) $\frac{\log N}{\log(a/b)} \geq \min\left\{1, \frac{\log m}{-\log a}, 2\left(1 - \frac{\log m}{-\log b}\right)\right\}$

Then

$$\dim_H(\Lambda) = \dim_H(\nu) = \min\left\{\frac{\log m}{-\log a}, 1 + \frac{\log(Na)}{-\log b}\right\}.$$

Recall that μ denotes the uniform Bernoulli measure on Σ and $\nu = \Pi_*\mu$ is its projection to the attractor. For a given $\theta \subset \mathbb{R}^2$ linear subspace let $proj_\theta$ denote the orthogonal projection to $\theta^\perp$.

Definition 4.5.3. *Let $\Phi = \{\phi_i\}_{i=1}^m$ be a strictly contracting IFS on $\mathbb{R}^d$. Let ϖ be a σ-invariant ergodic measure on Σ. We write $\mathcal{P} = \{[1], \dots, [m]\}$ for the partition on Σ defined by the first digits, and $\mathcal{B}$ for the Borel σ-algebra of $\mathbb{R}^d$. As Feng and Hu defined it in [6], the* ***Projection entropy*** *of ϖ under Π with respect to Φ is*

$$h_\Pi(\varpi) := H_\varpi(\mathcal{P}|\sigma^{-1}\Pi^{-1}\mathcal{B}) - H_\varpi(\mathcal{P}|\Pi^{-1}\mathcal{B}),$$

where $H_\varpi(\eta_1|\eta_2) = -\int \log(\varpi|\eta_2(x))(\eta_1(x))d\varpi(x)$ is the usual conditional entropy of η_1 given η_2 with respect to ϖ. Thus here $\varpi|\eta_2(x)$ is the conditional measure of ϖ supported on $\eta_2(x)$ where the latter denotes the element of the partition η_2 which contains x.

By Theorem 2.2 of [2]

$$\dim \nu = \frac{h_\Pi(\mu)}{-\log b} + \left(1 - \frac{-\log a}{-\log b}\right) \dim(proj_\theta)_*\nu \tag{4.10}$$

for $(\Pi_{BF})_*\mu$-almost every θ.

Lemma 4.5.1 ([3], Lemma 5.2)**.** *If (C1), (C2) and (C4) of Theorem 4.5.1 hold then $h_\Pi(\mu) = h(\mu) = \log m$, where $h(\mu)$ is the entropy of μ.*

Lemma 4.5.2 ([3], Lemma 5.3)**.** *Let $\mathcal{F}$ be a self-affine IFS of the form* (4.5) *satisfying the assumptions of Theorem 4.5.1. Then*

$$\dim \nu \geq \min\left\{2\frac{h(\mu)}{-\log b}, \frac{h(\mu)}{-\log a}, 1 + \frac{h(\mu) + \log a}{-\log b}\right\}.$$

Proof. By Theorem 1.1 of [7]

$$\dim(\Pi_{BF})_*\mu = \min\left\{1, \frac{h(\mu)}{\log a - \log b}\right\}. \tag{4.11}$$

By Lemma 4.3 of [1]

$$\dim(proj_\theta)_*\nu \geq \min\{\dim(\Pi_{BF})_*\mu, \dim\nu\}, \tag{4.12}$$

for $(\Pi_{BF})_*\mu$-almost every θ. We define the sequence $\{x_n\}_{n=0}^{\infty}$ inductively. Let $x_0 = h(\mu)/(-\log b)$ and $x_n = r(x_{n-1})$ for $n \geq 1$, where

$$r(x) = \frac{h(\mu)}{-\log b} + \left(1 - \frac{\log a}{\log b}\right) \min\left\{1, \frac{h(\mu)}{\log(a/b)}, x\right\}.$$

This r function is a contraction, thus iterating it will lead us to its fixed point.

$$\lim_{n\to\infty} x_n = \min\left\{2\frac{h(\mu)}{-\log b}, \frac{h(\mu)}{-\log a}, 1 + \frac{h(\mu) + \log a}{-\log b}\right\}.$$

Combine (4.10) with Lemma 4.5.1 and the formulas we gave for $\dim(proj_\theta)_*\nu$ and $\dim(\Pi_{BF})_*\mu$ to conclude by induction that $\dim\nu \geq x_n$ for every $n \geq 0$ as required. □

Now we can prove Theorem 4.5.1.

Proof. Observe that (4.11), Lemma 4.5.2 and $(C5)$ together implies

$$\dim(\Pi_{BF})_*\mu > \min\{\dim\nu, 2 - \dim\nu\}.$$

But then by Proposition 6.1 of [12] and (4.12)

$$\dim(proj_\theta)_*\nu = \min\{1, \dim\nu\}$$

for $(\Pi_{BF})_*\mu$-almost every θ. Thus by applying (4.10) and Lemma 4.5.1 the proof is finished. □

Chapter 5

An important application

In the last chapter we introduced the important notion of the Furstenberg measure. We will show how useful it is giving conditions which will provide that the affinity dimension of the attractor equals to the Hausdorff dimension following [4].

An IFS is called carpet if its affinities preserve the horizontal and vertical directions. It has been shown that the Hausdroff and box dimensions of a carpet depend on the projection of the set to the weak contracting direction. However a general self-affine system do not have an invariant contracting direction. Instead in an analogous fashion we can examine the typical dimension of the projections of the set in directions chosen according to the Furstenberg measure μ_F on the projective line $\mathbb{RP}^1$. In our case μ_F will be the projection of the Käenmäki measure which is supported by E and typically has Hausdorff dimension $\dim_H \mu = \dim_H E$.

5.1 Dynamics on projections

Consider the IFS $\{T_i x = A_i x + d_i\}_{i=1}^m$ with attractor E, where A_i are linear contractions on $\mathbb{R}^2$ and $d_i \in \mathbb{R}^2$ are translation vectors. We will assume that the SSP holds for this system and that $\forall i \in \{1, ..., m\} : A_i$ have strictly positive entries, that is the linear parts of the defining affine transformations map the first quadrant Q_1 into itself.

Rescaling does not affect the dimension hence we might assume that each T_i maps the unit disc D inside itself. For composition of functions and the image of E under such a composition we will use the standard notation:

$$T_{a_1...a_n} := T_{a_1}T_{a_2}...T_{a_n}, \quad E_{a_1...a_n} := T_{a_1...a_n}(E),$$

where $1 \le a_i \le m$ and $i \in \{1, ..., m\}$. Write $\alpha_1(a_1...a_n) \ge \alpha_2(a_1...a_n) > 0$ for the singular values of $A_{a_1...a_n}$. It geometrically means that the major and minor semiaxes of $D_{a_1...a_n}$ has length $\alpha_1(a_1...a_n)$ and $\alpha_2(a_1...a_n)$ respectively. Note that these values only dependent on the matrices A_i and not the d_i translation vectors.

Let μ be a measure on Σ. We will identify this measure with its projection under $\Pi : \Sigma \to E, \quad \Pi(a) = \lim_{n\to\infty} T_{a|n}(0)$ in the natural way, thus $\mu\,[a|n] = \mu(E_{a|n})$. Recall that the Lyapunov dimension of μ is given by

$$D(\mu) := \begin{cases} \frac{h(\mu)}{-\lambda_1(\mu)}, & \text{if } h(\mu) \le \lambda_1(\mu), \\ 1 + \frac{h(\mu)+\lambda_1(\mu)}{-\lambda_2(\mu)}, & \text{if } h(\mu) \ge \lambda_2(\mu), \end{cases}$$

where $h(\mu)$ is the Kolmogorov-Sinai entropy, and the Lyapunov exponents $\lambda_1(\mu)$, $\lambda_2(\mu)$ are provided by the Oseledec theorem. Note that $D(\mu)$ depends only on the A_i matrices and the choosen measure μ.

There exists an ergodic probability measure μ on Σ, called Käenmäki measure, which is shift invariant and satisfies $D(\mu) = \dim_A(A_1...A_m)$. This measure is a t-equilibrium measure and also a Gibbs measure given that the A_i matrices have strictly positive terms. Further μ will denote the Käenmäki measure.

Each defining matrix of our system induces a projective linear transformation $\phi_i : \mathbb{RP}^1 \to \mathbb{RP}^1$ on the projective line

$$\phi_i(\theta) := \frac{A_i^{-1}(\theta)}{\|A_i^{-1}(\theta)\|}, \quad 1 \le i \le m,$$

where $\|\ \|$ denotes the Euclidean norm, and $\mathbb{RP}^1$ is parametrized in the usual way. Consider the following stochastic process on the projective line: Start at an arbitrary point and in each step use ϕ_i on it, where i is chosen according to μ. According to the note after Definition 4.4.1 in the previous chapter, the stationary measure of this process will be the Furstenberg measure μ_F according to μ. We could also define μ_F as the weak limit for μ almost all $a \in \Sigma$ and all $\theta \in \mathbb{RP}^1$ of the sequence of measures

$$\frac{1}{n}\sum_{k=0}^{n-1} \delta_{\phi_{a_k \dots a_1}}(\theta),$$

where $\phi_{a_n \dots a_1} := \phi_{a_n}\phi_{a_{n-1}} \dots \phi_{a_1}$.

All matrices A_i has strictly positive entries thus the transformations ϕ_i are strict contractions of the negative quadrant $\mathcal{Q}_2 \subset \mathbb{RP}^1$ under a metric $d(\theta_1, \theta_2)$ given by the absolute angle between $\theta_1, \theta_2 \in \mathcal{Q}_2$. Thus $\{\phi_1, ..., \phi_m\}$ is an IFS, and the Furstenberg measure is an invariant probability measure of this system.

For any $\theta \in \mathbb{RP}^1$ let $\pi_\theta : E \to [-1, 1]$ be the projection of the attractor of our original system E onto the $\theta^\perp$ angle diameter of the unit disc D where this diameter is isometrically identified with $[-1, 1]$. We also define the projection $\Pi_\theta : \Sigma \to [-1, 1]$, $\Pi_\theta := \pi_\theta \circ \pi$ and the projected measure $\mu_\theta := \mu \circ \Pi_\theta^{-1}$ for every $\theta \in \mathbb{RP}^1$.

From now on we will work on the two-sided shift space $\Sigma^\pm := \{1, ..., m\}^{\mathbb{Z}}$, and we will write $\overline{a}$ for a typical member $\dots a_{-2}a_{-1}a_0a_1a_2 \dots$. We define $\overline{\mu}$ to be the unique, shift invariant measure on $\Sigma^\pm$ for which $\mu\,[a_m...a_n] = \overline{\mu}\,[a_m...a_n]$ for every cylinder which only dependent of positive coordinates. The functions ϕ_i are contractions thus the limit

$$\rho(\overline{a}) := \lim_{n\to\infty} \phi_{a_0}\phi_{a_{-1}} \dots \phi_{a_{-n}}(0)$$

exists for all $\overline{a} \in \Sigma^\pm$. We define the map $P : \Sigma^\pm \to \mathbb{RP}^1 \times \Sigma$ by $P(\overline{a}) := (\rho(\overline{a}), a_1a_2 \dots)$. Thus with the help of the function P we can assign an angle in $\mathcal{Q}_2$ and an $a \in \Sigma$ to any $\overline{a} \in \Sigma^\pm$. Let ν be the push-forward measure of $\overline{\mu}$ by P

$$\nu := \overline{\mu} \circ P^{-1}$$

which is defined on $\mathcal{Q}_2 \times \Sigma$.

Lemma 5.1.1 ([4], Lemma 3.1). *The map $P \circ \sigma \circ P^{-1} : \mathbb{RP}^1 \times \Sigma \to \mathbb{RP}^1 \times \Sigma$ is well defined, and the system $(\mathbb{RP}^1 \times \Sigma, \nu, P \circ \sigma P^{-1})$ is ergodic.*

Proof. Given $(\theta, a) \in \mathbb{RP}^1 \times \Sigma$ choose an $\overline{a} \in P^{-1}(\theta, a)$. Then

$$\rho(\sigma\overline{a}) = \lim_{n\to\infty} \phi_{a_1} \circ \phi_{a_0} \circ \phi_{a_{-1}} \circ \cdots \circ \phi_{a_{-n}}(0) = \phi_{a_1}(\rho(\overline{a})) = \phi_{a_1}(\theta),$$

which means that

$$P \circ \sigma \circ P^{-1}(\theta, a) = (\phi_{a_1}(\theta), \sigma(a)).$$

One can see that if $\overline{a}, \overline{a}' \in P^{-1}(\theta, a)$, then $P(\sigma\overline{a}) = P(\sigma\overline{a}')$ thus $P \circ \sigma \circ P^{-1}$ is well defined.

The system $(\mathbb{RP}^1 \times \Sigma, \nu, P \circ \sigma \circ P^{-1})$ is a factor of the ergodic system $(\Sigma^{\pm}, \overline{\mu}, \sigma)$ under the map P. Using that the factors of an ergodic system are also ergodic the result follows. □

It must be emphasized that for Bernoulli measures the "past" $\dots a_{-2}a_{-1}a_0$ and the "future" $a_1a_2\dots$ are independent but for a general $\overline{\mu}$ it will not hold necessarily. Fortunately our $\overline{\mu}$ is originated from the Käenmäki measure thus it is a Gibbs measure. Because of that it has the quasi-Bernoulli property, that is there exists a constant $C > 0$ such that

$$\frac{1}{C}\overline{\mu}\left[a_{-n}\dots a_k\right] \le \overline{\mu}\left[a_{-n}\dots a_0\right]\overline{\mu}\left[a_1\dots a_k\right] \le C\overline{\mu}\left[a_{-n}\dots a_k\right].$$

Thus $\overline{\mu}$ is equivalent to the measure $\tilde{\mu}$ on $\Sigma^{\pm}$ given by

$$\tilde{\mu}\left[a_{-n}\dots a_k\right] = \overline{\mu}\left[a_{-n}\dots a_0\right]\overline{\mu}\left[a_1\dots a_k\right]$$

which is not invariant but has independent past and future. After projecting it with P we get that $\nu = \overline{\mu} \circ P^{-1}$ is equivalent to the product measure $\tilde{\nu} = \tilde{\mu} \circ P^{-1} = \mu_F \times \mu$ on $\mathbb{RP}^1 \times \Sigma$. This measure will be useful later since a product measure is easier to work with.

If E was a self-affine set with an invariant strong contracting foliation like a carpet, then the projection of E parallel to the strong contracting direction would be self-similar sets. In our case the contracting directions are not invariant but we can express the projections of E in terms of projections to other directions.

Lemma 5.1.2 ([4], Lemma 3.2)**.** *For each $\theta \in \mathbb{RP}^1$ and $i \in \{1, ..., m\}$ the map $f_{i,\theta} : [-1, 1] \to [-1, 1]$ given by*

$$f_{i,\theta} = \pi_\theta \circ T_i \circ \pi^{-1}_{\phi_i(\theta)} \tag{5.1}$$

is well-defined, and $\pi_\theta(T_i(E)) = f_{i,\theta}(\pi_{\phi_i(\theta)}(E))$.

Proof. Let us write l_θ for the line which goes through the origin and has angle θ. By definition $\pi_{\phi_i(\theta)}$ projects onto $l_{\phi_i(\theta)^\perp}$, thus $\pi^{-1}_{\phi_i(\theta)}(x)$ is a line parallel to $l_{\phi_i(\theta)}$ for any $x \in [-1, 1]$. We know from the definition of ϕ_i that $T_i \circ \pi^{-1}_{\phi_i(\theta)}(x)$ is parallel to l_θ. All the points of such a line will be projected into the same point by π_θ thus $f_{i,\theta}$ is well defined and also affine, because of the affinity of T_i.

Following the path of an arbitrary x in $\pi_\theta(T_i(E))$ we observe

$$x \in \pi_\theta(T_i(E)) \Longleftrightarrow \pi_\theta^{-1}(x) \cap T_i(E) \neq \emptyset \Longrightarrow T_i^{-1}(\pi_\theta^{-1}(x)) \cap E \neq \emptyset,$$

and by the definition of $f_{i,\theta}$

$$f^{-1}_{i,\theta}(x) = \pi_{\phi_i(\theta)}(T_i^{-1}(\pi_\theta^{-1}(x))) \in \pi_{\phi_i(\theta)}(E).$$

Thus $\pi_\theta(T_i(E)) = f_{i,\theta}(\pi_{\phi_i(\theta)}(E))$ since $f_{i,\theta}$ is affine. □

This lemma means that the projections of E are acting almost like a self-affine family in the following sense.

Proposition 5.1.1 ([4], Proposition 3.3)**.** *Define $f_{i,\theta}$ as (5.1), then*

$$\pi_\theta(E) = \cup_{i=1}^m f_{i,\theta}(\pi_{\phi_i(\theta)}(E)) \tag{5.2}$$

holds for all $\theta \in \mathbb{RP}^1$.

Proof. By using that E is a self-affine set and lemma 5.1.2 we get

$$\pi_\theta(E) = \cup_{i=1}^m \pi_\theta(T_i(E)) = \cup_{i=1}^m f_{i,\theta}(\pi_{\phi_i(\theta)}(E)).$$

□

The next corollary not just uses this relation between the projections of E, but also shows how useful can the Furstenberg measure be if we want to word a statement of this kind.

Corollary 5.1.1 ([4], Corollary 3.4)**.** *The dimensions* $\dim_H \pi_\theta(E), \underline{\dim}_B \pi_\theta(E)$ *and* $\overline{\dim}_B \pi_\theta(E)$ *are each constant for almost all* $\theta \in \mathbb{RP}^1$ *with respect to* μ_f*.*

Proof. It follows from (5.2) that $\dim_H \pi_\theta(E) \geq \dim_H \pi_{\phi_i(\theta)}(E)$ for all $\theta \in \mathbb{RP}^1$, $1 \leq i \leq m$, and the same holds for the other dimensions as well. Since our Furstenberg measure μ_F is ergodic the result follows for μ_F-almost all $\theta \in \mathbb{RP}^1$. □

5.2 Equality of dimensions

We want to conclude this chapter by the proof of the following theorem.

Theorem 5.2.1 ([4], Theorem 1.1)**.** *Let* $E \subset \mathbb{R}^2$ *be the attractor of the IFS* $\{A_i x + d_i\}_{i=1}^m$*, where the* A_i *are strictly positive matrices, and assume that the strong separation property holds for this system. Let* μ *be the Käenmäki measure on the symbolic space and* μ_F *the corresponding Furstenberg measure.*

If the projection of μ *in the direction* θ *is absolutely continuous for* μ_F*-almost every* θ*, then the measure* μ *is exact dimensional and* $\dim_H E = \dim_B E = \dim_A E$*.*

We want to give a lower estimate on the Hausdorff dimension using the local dimension of μ which we can relate to the local dimension of the projections of μ. The next lemma help us estimate the local dimension of these images by comparing the measures of small balls with the projected measures of intervals.

Note that since the A_i are linear and map the first quadrant strictly into its interior we can find $\lambda < 1$ so that each A_i contract angles between lines in the first quadrant by λ or less. The image $A_{i_1}...A_{i_n}(R)$ of the unit square R is a parallelogram, and thus one of its angles is at most $(\pi/2)\lambda^n$. By trigonometry, the ratio of the width to the diameter of that parallelogram is bigger than $\alpha_2(a|_n)/\alpha_1(a|_n)$ because of the definition of the singular values, but smaller or equal than $(\pi/2)\lambda^n$.

Lemma 5.2.1 ([4], Lemma 4.2). *Let $V \subset \mathcal{Q}_2$ be a strict subset for which $\phi_i : V \to \int V$ holds for all i. Then there exist numbers $C > 0$, $0 < \rho_1 < \rho_2$ such that for each $a \in \Sigma$, $n \in \mathbb{N}$ and $\theta \in V$:*

$$
\begin{aligned}
&C^{-1}\mu(B(\pi(a), \rho_1\alpha_2(a|_n))) \\
&\quad \leq \mu\left[a|_n\right]\mu_{\phi_{a_n \dots a_1}(\theta)}\left[\pi_{\phi_{a_n \dots a_1}(\theta)}(\sigma^n(a)) - \frac{\alpha_2(a|_n)}{\alpha_1(a|_n)}, \pi_{\phi_{a_n \dots a_1}(\theta)}(\sigma^n(a)) + \frac{\alpha_2(a|_n)}{\alpha_1(a|_n)}\right] \\
&\quad \leq C\mu(B(\pi(a), \rho_2\alpha_2(a|_n))).
\end{aligned}
$$

This lemma shows that if we want to estimate the local dimension of μ at a we need only estimate the local dimension of the projected measure $\mu_{\phi_{a_n \dots a_1}(\theta)}$ at $\pi_{\phi_{a_n \dots a_1}(\theta)}(\sigma^n(a))$. Thus we can work with the approximate local dimensions of Lemma 5.2.1, write

$$d(\theta, a, n) := \frac{\log \mu_{\phi_{a_n \dots a_1}(\theta)}(B(\pi_{\phi_{a_n \dots a_1}(\theta)}(\sigma^n(a)), \alpha_2(a|_n)/\alpha_1(a|_n)))}{\log(\alpha_2(a|_n)/\alpha_1(a|_n))}.$$

Lemma 5.2.2 ([4], Lemma 4.3). *For ν-almost every pair $(\theta, a) \in \mathbb{PR}^1 \times \Sigma$ and $\forall \epsilon > 0$ we define*

$$
\begin{aligned}
&G(\theta, a, \epsilon) := \{n \in \mathbb{N} : |d(\theta, a, n) - 1| < \epsilon\} \\
&G_N(\theta, a, \epsilon) := \{n \in \mathbb{N} : |d(\theta, a, n) - 1| < \epsilon,\ 1 \leq n \leq N\} = G(\theta, a, \epsilon) \cap \{1, \dots, N\}.
\end{aligned}
$$

Then $\lim_{n\to\infty} |G_N(\theta, a, \epsilon)|/N = 1$.

With the help of this lemma we can prove the following proposition that let us approximate the local dimension of the measure μ using its Lyapunov dimension $D(\mu)$.

Proposition 5.2.1 ([4], Proposition 4.4). *For μ almost all $a \in \Sigma$ and for all $\epsilon > 0$, we have*

$$\lim_{N\to\infty} \frac{1}{N}\left|\left\{n \in \{1, ..., N\} : \left|\frac{\log \mu(B(\pi(a), \alpha_2(a|n)))}{\log(\alpha_2(a|_n))} - D(\mu)\right| > \epsilon\right\}\right| = 0, \tag{5.3}$$

where $D(\mu)$ is the Lyapunov dimension of μ.

Proof. Recall that by the Shannon-McMillan-Breiman theorem

$$\lim_{n\to\infty} -\frac{1}{n}\log\mu\,[a|_n] = h(\mu),$$

thus by using Oseledec's theorem

$$\lim_{n\to\infty}\frac{\log\mu\,[a|_n]}{\log\alpha_2(a|_n)} = \lim_{n\to\infty}\frac{\log\mu\,[a|_n]}{n}\left(\frac{\log\alpha_2(a|_n)}{n}\right)^{-1} = \frac{-h(\mu)}{\lambda_\mu^2}, \tag{5.4}$$

for μ-almost every $a \in \Sigma$. If we use the left hand inequality of Lemma 5.2.1 and a little algebra we obtain for μ-almost every $a \in \Sigma$, μ_F-almost every $\theta \in V$ and for all $n \in G(\theta, a, \epsilon)$

$$\begin{aligned}
&\frac{\log\mu(B(\pi(a),\rho_1\alpha_2(a|_n)))}{\log(\rho_1\alpha_2(a|_n))}\\
&\le \frac{\log(C\mu\,[a|_n]\,\mu_{\phi_{a_n\ldots a_1}(\theta)}(B(\pi_{\phi_{a_n\ldots a_1}(\theta)}(\sigma^n(a)),\alpha_2(a|_n)/\alpha_1(a|_n))))}{\log(\rho_1\alpha_2(a|_n))}\\
&= \frac{\log(C\mu\,[a|_n])}{\log(\rho_1\alpha_2(a|_n))} + \frac{\log(\mu_{\phi_{a_n\ldots a_1}(\theta)}(B(\pi_{\phi_{a_n\ldots a_1}(\theta)}(\sigma^n(a)),\alpha_2(a|_n)/\alpha_1(a|_n))))}{\log(\rho_1\alpha_2(a|_n))}\\
&= \frac{\log(C\mu\,[a|_n])}{\log(\rho_1\alpha_2(a|_n))} + d(\theta,a,\epsilon)\cdot\frac{\log(\alpha_2(a|_n)/\alpha_1(a|_n))}{\log(\rho_1\alpha_2(a|_n))}\\
&\le \frac{-h(\mu)}{\lambda_2(\mu)} + (1+\epsilon)\cdot\frac{\lambda_2(\mu)-\lambda_1(\mu)}{\lambda_2(\mu)}\\
&= \frac{h(\mu)+(1+\epsilon)\lambda_1(\mu)}{-\lambda_2(\mu)} + 1 + \epsilon \le (1+\epsilon)D(\mu)+\epsilon.
\end{aligned}$$

We have used (5.4), Oseledec's multiplicative ergodic theorem, the definition of the Lyapunov dimension and that quotients in the fourth row are bounded from above by their limits. We know from Lemma 5.2.2 that for ν-almost all $(\theta, a) \in \mathbb{RP}^1 \times \Sigma$ the set $G(\theta, a, \epsilon)$ has density 1 for all $\epsilon > 0$, thus we can conclude that

$$\frac{\log\mu(B(\pi(a),\rho_1\alpha_2(a|_n)))}{\log(\rho_1\alpha_2(a|_n))} \le (1+\epsilon)D(\mu)+\epsilon$$

on a set of n of density 1 for μ-almost every a.

A similar reverse inequality follows from the right side of Lemma 5.2.1 with ρ_2 instead of ρ_1. Since $\alpha_2(a|_n) \to 0$ the constants ρ_1, ρ_2 are irrelevant if we take the limit as $n \to \infty$. The statement of the proposition follows, since we chose ϵ to be arbitrary small.

$\square$

Note that the upper and lower limits of $\log \mu(B(x,r))/\log r$ as $r \to 0$ are coincide with the limits of $\log \mu(B(x,r_k))/\log r_k$ as $k \to \infty$, if $r_k \searrow 0$ and $\log r_{k+1}/\log r_k \to 1$. Thus we can calculate the local dimension of μ by substituting $r_k = \rho_1 \alpha_2(a|_{n_k})$ into this limit, where n_k is any increasing sequence of positive integers of density 1. It follows from Proposition 5.2.1 that for μ-almost all $a \in \Sigma$ and $\forall \epsilon > 0$,

$$\left| \frac{\log \mu(B(\pi(a),))}{\log r} - D(\mu) \right| < 2\epsilon$$

if r is small enough. Thus the local dimension of μ exists and is equal to the Lyapunov dimension $D(\mu)$ at μ-almost all a.

We chose μ to be the Käenmäki measure, therefore it is supported by E and satisfies $\dim_A(A_1, ..., A_m) = D(\mu)$. We know that $\dim_H E \leq \dim_A E$ for all self-affine sets, and we just showed that under the conditions of Theorem 5.2.1

$$\dim_A(A_1, ..., A_m) = D(\mu) \leq \dim_H E$$

which proves the Theorem.

Chapter 6

A more general case

Instead of showing how the Furstenberg measure works, in this chapter we show that it exists even in a more general setting. We will work with 2×2 contractive matrices of real terms $\{A_i\}_{i=1}^m$. We follow [11] and define the following family of matrices.

Definition 6.0.1. *We call a 2×2 matrix **row allowable** if every row contains a non-zero element. We write $\mathcal{RA}$ for the set of row allowable matrices.*

Further write $\mathcal{M}^+$ for the set of 2×2 non-singular matrices which only has non-negative terms. Each element of $\mathcal{M}^+$ is row allowable and must fall into one of these three types:

$$\mathcal{T}_1 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d > 0 \right\},$$
$$\mathcal{T}_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, d > 0; b, c \geq 0 \right\},$$
$$\mathcal{T}_3 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : b, c > 0; a, d \geq 0 \right\}.$$

For the vectors that only have positive terms $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2) > \mathbf{0}$ we define the following function called **projective distance**

$$d(\mathbf{x}, \mathbf{y}) := \log \left[\frac{\max_i (x_i / y_i)}{\min_j (x_j / y_j)} \right].$$

If $\mathbf{x} = \lambda \mathbf{y}$ for some $\lambda \in \mathbb{R}$ then $d(\mathbf{x}, \mathbf{y}) = 0$, hence it is only a pseudo-metric. However d is a metric on

$$\Delta := \left\{ \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2 : x_i > 0 \text{ and } \sum_{i=1}^{2} x_i = 1 \right\}.$$

For all $A \in \mathcal{RA}$ we define

$$\tilde{A} : \Delta \to \Delta \quad \tilde{A}(\mathbf{x}) := \frac{A\mathbf{x}}{\|A\mathbf{x}\|_1}.$$

Let $\tau(A)$ denote the Lipschitz constant of any $A \in \mathcal{RA}$, particularly

$$\tau(A) := \sup_{\mathbf{x}, \mathbf{y} \in \Delta, \mathbf{x} \neq \mathbf{y}} \frac{d(A\mathbf{x}, A\mathbf{y})}{d(\mathbf{x}, \mathbf{y})},$$

and write $Lip(\alpha)$ for the set of functions with Lipschitz constant $0 < \alpha$.

On the complete metric space (Δ, d) we write $M(\Delta)$ for the set of all probability measures μ on Δ for which $\mu(\phi) < \infty$ holds for all real-valued Lipschitz functions ϕ defined on (Δ, d). The measure of a function is as usual

$$\mu(\phi) := \int \phi(\mathbf{x}) \mu(\mathbf{x}).$$

We define the distance of $\mu, \nu \in M(\Delta)$ by

$$L(\mu, \nu) := \sup\{\mu(\phi) - \nu(\phi) | \phi : \Delta \to \mathbb{R}, \tau(\phi) \leq 1\}.$$

We want to show that for a given probability vector $\mathbf{p} = (p_1, \ldots, p_m)$ there exists a measure $\nu \in M(\Delta)$ such that

$$\nu(H) = \sum_{i=1}^{m} \nu(\tilde{A}_i^{-1}(H))$$

for any $H \subset \Delta$. This measure would be the Furstenberg measure, and this definition coincides with the one involving a Markov chain on the projective line mentioned in the previous chapter, and in the note right after Definition 4.4.1.

We define the following operator $\mathcal{F} : M(\Delta) \to M(\Delta)$

$$\nu \mapsto \mathcal{F}(\nu), \quad \forall H \subset \Delta : (\mathcal{F}(\nu))(H) := \sum_{i=1}^{m} p_i \nu(\tilde{A}_i^{-1} H),$$

or equivalently for Lipschitz continuous maps

$$(\mathcal{F}(\nu))(\phi) := \sum_{i=1}^{m} p_i \nu(\phi(\tilde{A}_i \mathbf{x})) = \sum_{i=1}^{m} p_i \int (\phi \circ \tilde{A}_i)(\mathbf{x}) d\nu(\mathbf{x}).$$

Theorem 6.0.1. *If $A_1 \in \mathcal{T}_1$ and $A_2, \ldots A_m \in \cup_{i=1}^{3} \mathcal{T}_i$, then $\mathcal{F}$ is a contraction on $M(\Delta)$.*

Proof. Since $\forall i \in \{1, \ldots, m\} : \tilde{A}_i$ is a contraction for any ϕ with $Lip(\phi) \leq 1$ we have

$$\phi \circ \tilde{A}_i \in Lip(\tau(\tilde{A}_i)) \Rightarrow \tau(\phi \circ \tilde{A}_i) \leq \tau(\tilde{A}_i) \leq 1.$$

We know that $A_1 \in \mathcal{T}_1$ thus $\tau(A_1) < 1$ by [13, Theorem 3.10] and [13, Theorem 3.12]. However a given matrix $B \in \mathcal{T}_2$ might have the form

$$B = \begin{bmatrix} b & 0 \\ 0 & b \end{bmatrix},$$

with some $b \in (0, 1)$, and for such a matrix $\tau(B) = 1$. Similarly, there are matrices in $\mathcal{T}_3$ with Lipschitz constant 1.

Let μ and ν be distinct measures. Using the definition of $L(\mu, \nu)$ we obtain

$$\begin{aligned}(\mathcal{F}(\mu))(\phi) - (\mathcal{F}(\nu))(\phi) &= \sum_{i=1}^{m} p_i \left(\int (\phi \circ \tilde{A}_i)(\mathbf{x}) d\mu(\mathbf{x}) - \int (\phi \circ \tilde{A}_i)(\mathbf{x}) d\nu(\mathbf{x}) \right) \\ &\leq p_1 \cdot L(\mu, \nu) \cdot \tau(A_1) + \sum_{i=2}^{m} p_i \cdot L(\mu, \nu) \\ &= L(\mu, \nu)(p_1 \cdot \tau(A_1) + p_2 + \cdots + p_m) < L(\mu, \nu).\end{aligned}$$

□

Let μ be the Bernoulli measure on $\Sigma = \{1, \dots m\}^{\mathbb{N}}$ induced by the probability vector $\mathbf{p} = (p_1, \dots, p_m)$. For a μ-typical $\mathbf{i} \in \Sigma$ we have $\tilde{A}_{\mathbf{i}|n}(\Delta) \subset K_{\mathbf{i}|_n} \subset \Delta$ if n is big enough, where $K_{\mathbf{i}|_n}$ is a compact subset of Δ. This implies that as $n \to \infty$, $\tilde{A}_{\mathbf{i}|_n}$ contracts Δ into a point for μ-almost all $\mathbf{i}$. Thus the angle $\lim_{n\to\infty} \alpha^F_{\mathbf{i}|_n}$ what we defined in Chapter 4.2 on page 25 exists in limes for μ-almost all $\mathbf{i}$.

If the system $\{\tilde{A}_i\}_{i=1}^m$ satisfies the assumption of Theorem 6.0.1, then $\mathcal{F}$ is a contraction. Therefore it has a unique fixed point $\nu \in M(\Delta)$. Let $\tilde{\Pi} : \Sigma \to \Delta$ be the natural projection of the system $\{\tilde{A}_i\}_{i=1}^m$. According to Definition 4.1.1 the Furstenberg measure in this case is $\mu_F := \tilde{\Pi}_*\mu$, which is also a fixed point of $\mathcal{F}$. It follows from this reasoning that the Furstenberg measure can be defined as the unique fixed point of $\mathcal{F}$.

6.1 Connection with Lyapunov exponents

From now on we assume that $\forall i : A_i \in \mathcal{T}_1$, thus all the matrices are strict contractions on Δ, and fix an $\mathbf{i} \in \Sigma$. It implies that for all $\mathbf{u} \in \Delta$ there exists a δ such that

$$\|A_{i_1 \dots i_n}\mathbf{u}\|_1 = \|A_{i_1 \dots i_{n-1}}(A_{i_n}\mathbf{u})\|_1 \geq \delta \cdot \|A_{i_1 \dots i_{n-1}}\|_1 \tag{6.1}$$

holds for any n. Write L for the maximal term of A_{i_n} to obtain

$$\|A_{i_1 \dots i_n}\|_1 = \|A_{i_1 \dots i_{n-1}}A_{i_n}\|_1 \leq 2L \cdot \|A_{i_1 \dots i_{n-1}}\|_1. \tag{6.2}$$

Let $\mathbf{v}(\mathbf{i}) := \tilde{\Pi}(\mathbf{i})$ denote the vector on Δ which is coded by $\mathbf{i}$. The definition of the norm combined with (6.2) and (6.1) yields

$$\|A_{i_1 \dots i_n}\|_1 \geq \|A_{i_1 \dots i_n}\mathbf{v}(\sigma^n \mathbf{i})\| \geq c \cdot \|A_{i_1 \dots i_n}\|_1, \tag{6.3}$$

for some constant $c := \delta/2L$, with σ being the left shift on Σ as usual.

Since $\|\mathbf{v}(\sigma^k\mathbf{i})\|_1 = 1$ for all k and all $\mathbf{i}$ we can write

$$\begin{aligned}
\|A_{i_1\dots i_n}\mathbf{v}(\sigma^n\mathbf{i})\|_1 &= \frac{\|A_{i_1\dots i_n}\mathbf{v}(\sigma^n\mathbf{i})\|_1}{\|\mathbf{v}(\sigma^n\mathbf{i})\|_1} \\
&= \frac{\|A_{i_1\dots i_n}\mathbf{v}(\sigma^n\mathbf{i})\|_1}{\|A_{i_2\dots i_n}\mathbf{v}(\sigma^n\mathbf{i})\|_1} \cdot \frac{\|A_{i_2\dots i_n}\mathbf{v}(\sigma^n\mathbf{i})\|_1}{\|A_{i_3\dots i_n}\mathbf{v}(\sigma^n\mathbf{i})\|_1} \cdots \frac{\|A_{i_n}\mathbf{v}(\sigma^n\mathbf{i})\|_1}{\|\mathbf{v}(\sigma^n\mathbf{i})\|_1} \\
&= \frac{\|A_{i_1}\mathbf{v}(\sigma\mathbf{i})\|_1}{\|\mathbf{v}(\sigma\mathbf{i})\|_1} \cdot \frac{\|A_{i_2}\mathbf{v}(\sigma^2\mathbf{i})\|_1}{\|\mathbf{v}(\sigma^2\mathbf{i})\|_1} \cdots \frac{\|A_{i_n}\mathbf{v}(\sigma^n\mathbf{i})\|_1}{\|\mathbf{v}(\sigma^n\mathbf{i})\|_1} \\
&= \prod_{k=1}^{n} \|A_{i_k}\mathbf{v}(\sigma^k\mathbf{i})\|_1.
\end{aligned}$$

After applying the above observation to (6.3) we obtain

$$\lim_{n\to\infty}\frac{1}{n}\log\|A_{i_1\dots i_n}\|_1 = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\log\|A_{i_k}\mathbf{v}(\sigma^k\mathbf{i})\|_1 = \int_\Sigma \log\|A_{i_1}\mathbf{v}(\sigma\mathbf{i})\|_1 d\mu(\mathbf{i}),$$

where in the last equation we used Birkhoff's ergodic theorem.

We can rewrite this integral in the following way

$$\begin{aligned}
\int_\Sigma \log\|A_{i_1}\mathbf{v}(\sigma\mathbf{i})\|_1 d\mu(\mathbf{i}) &= \sum_{k=1}^{n} p_k \int_\Sigma \log\|A_k\mathbf{v}(\mathbf{i})\|_1 d\mu(\mathbf{i}) \\
&= \sum_{k=1}^{n} p_k \int_\Delta \log\|A_k\mathbf{u}\|_1 d\mu_F(\mathbf{u})
\end{aligned}$$

Thus we have a relation between the Lyapunov exponents and the Furstenberg measure, namely

$$\lim_{n\to\infty}\frac{1}{n}\log\|A_{i_1\dots i_n}\|_1 = \sum_{k=1}^{n} p_k \int_\Delta \log\|A_k\mathbf{u}\|_1 d\mu_F(\mathbf{u}).$$

More general results for systems satisfying the so-called dominated splitting condition are presented in [1] and [2].

Chapter 7

Conclusion

This thesis can be separated into two distinct parts, both investigating the dimension theory of non-conformal sets. The system consisted in the first part is an IFS on the line which can be obtained in the following way: Consider a self-similar IFS on $[0, 1]$ for which the graph of the functions are straight line segments. At the fixed points of some of these affine functions we "break" this straight line and then continue it with a different slope. The graph of such a system is presented on Figure 3.2. We extended the theory of conformal IFS with this non-conformal family, and using the notion of graph directed iterated function systems we gave a formula for the Hausdorff dimension of these systems.

In the second part we collected some of the most important tools which have been introduced recently to study the theory of self-affine fractals. This theory went through a very intense development during the last 5 years. One of the reasons behind it was the introduction of the Furstenberg measure. In this thesis I presented some examples from the recent literature which show how to apply the Furstenberg measure in the dimension theory of self-affine fractals.

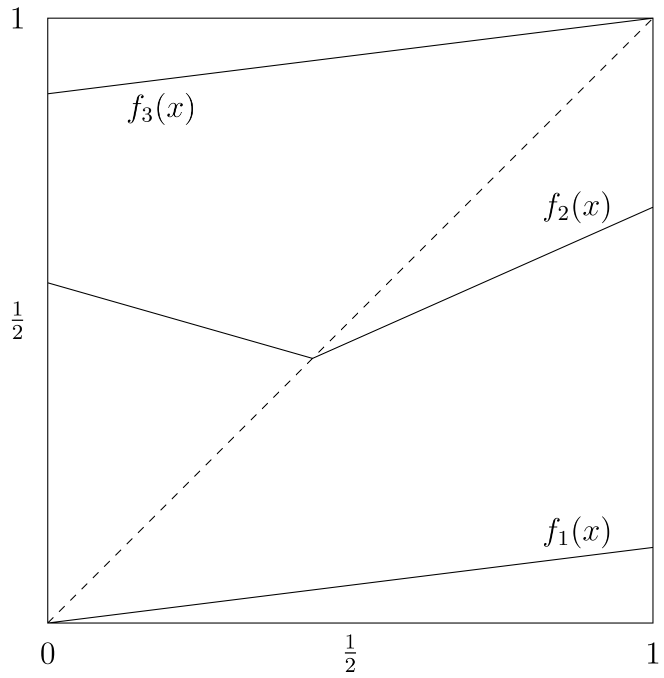

Figure 7.1: The graph of a non-monotonous broken IFS.

Further direction of research

I would like to extend the result obtained about the Broken IFS to the case when some of the functions are neither monotonously increasing nor monotonously decreasing. This case is much more difficult than the one treated in the thesis because of the heavy overlaps caused by the non-monotonic maps. The graph of such a system is shown on Figure 7.1.

I would also like to work on the theory of self-affine fractals and investigate the absolute continuity of the Furstenberg measure.

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