Limit Large Dev. Thms. Exam, June 7, 2024.

Info: Each of the 4 questions is worth 25 marks. Write your name and Neptun code on each piece of paper that you submit. Separate the solutions of different exercises (and sub-exercises) with a horizontal line. No calculators or electronic devices are allowed. No formula sheets are allowed. You have 90 minutes to complete this exam. You need to collect at least 40 points on this exam in order to pass the course.

1. (a) Define the lower convex envelope f^{co} and the Legendre transform $\widehat{f}$ of $f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$.

   (b) Show that $\widehat{f}$ is convex.

   (c) Show that if $f \le g$ then $\widehat{f} \ge \widehat{g}$.

   (d) Show that $\widehat{\widehat{f}} \le f^{co}$.

   (e) Show that $\widehat{\widehat{f}} \ge f^{co}$.

2. Let (X_n) denote a simple symmetric random walk on $\mathbb{Z}$. Let π_n denote amount of time the walker spends on the positive half-line up to time n. Let $u(k) = \mathbb{P}(X_k = 0)$. Show that

$$\mathbb{P}(\pi_{2n} = 2k) = u(2k) \cdot u(2 \cdot (n-k)), \qquad k = 0, 1, \ldots, n$$

   Instruction: You may use without proof that $\mathbb{P}(\pi_{2n} = 0) = u(2n)$.

3. Suppose that there is an urn with n different coupons in it. You start to draw coupons from the urn with replacement. In each round you pick each coupon with equal probability. Denote by V_n the number of coupons that you need to draw until you can say that you have touched all of the coupons at least once.

   Find $\lim_{n\to\infty} \mathbb{E}\left(\frac{V_n - n\ln(n)}{n}\right)$ and $\lim_{n\to\infty} \text{Var}\left(\frac{V_n - n\ln(n)}{n}\right)$.

   Hint: You may use without proof that if $X \sim \text{OPTGEO}(p)$ then $\mathbb{E}(X) = \frac{1}{p}$ and $\text{Var}(X) = \frac{1-p}{p^2}$.

4. Let X denote a random variable and let φ denote its characteristic function. Prove that for any $K > 0$ we have

$$\mathbb{P}(|X| \ge K) \le \frac{K}{2} \int_{-2/K}^{2/K} (1 - \varphi(t)) \, dt$$