Exercises to be solved in class, Stochastic Analysis, 2023 spring

1. Show that if X and Y are both simple random variables, moreover X and Y are independent, then

$$\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y).$$

2. Let (X, Y) denote a pair of simple random variables defined on the same probability space. Denote by $\{x_1, \dots, x_n\}$ the set of possible values of X and denote by $\{y_1, \dots, y_m\}$ the set of possible values of Y. Let $p_{i,j} = \mathbb{P}(X = x_i, Y = y_j)$ and $p_i = \sum_{j=1}^m p_{i,j}$. Denote by

$$H(x_i) := \sum_{j=1}^m y_j \frac{p_{i,j}}{p_i}, \quad i = 1, \dots, n.$$

   (a) Let $A_i = \{X = x_i\}$, $i = 1, \dots, n$. Show that the sigma-algebra $\sigma(X)$ generated by X consists of the events of form $A_I = \cup_{i \in I} A_i$, where $I \subseteq \{1, \dots, n\}$.

   (b) Prove that the random variable $H(X)$ satisfies the abstract definition of $\mathbb{E}(\, Y \,|\, \sigma(X)\,)$.
   Hint: You have to check that $Z := H(X)$ satisfies $\mathbb{E}(|H(X)|) < +\infty$, $H(X)$ is $\sigma(X)$-measurable and that $\mathbb{E}(Y \mathbb{1}_A) = \mathbb{E}(H(X)\mathbb{1}_A)$ for any $A \in \sigma(X)$.

3. Let (S_n) denote a 1-dimensional simple symmetric random walk. That is:

$$S_n = \eta_1 + \eta_2 + \cdots + \eta_n,$$

   where $\eta_1, \eta_2, \dots, \eta_n$ are independent and identically distributed *Rademacher* random variables:

$$\mathbb{P}(\eta_i = 1) = \mathbb{P}(\eta_i = -1) = \frac{1}{2}.$$

   Let $\mathcal{F}_n = \sigma(\eta_1, \dots, \eta_n) = \sigma(S_1, \dots, S_n)$, thus $(\mathcal{F}_n)$ is the filtration generated by (S_n).

   (a) Show that (S_n) is a martingale. (You will have to use some of the properties listed in the lecture notes – name them when you use them)

   (b) Show that if $\lambda \in \mathbb{R}$ and
$$M_n = e^{\lambda S_n - n \ln(\cosh(\lambda))},$$
   then (M_n) is a martingale. (Again, name the properties that you use)
   Hint: What is the moment generating function of a Rademacher random variable?

4. Let (S_n) denote a 1-dimensional simple symmetric random walk. That is:

$$S_n = \eta_1 + \eta_2 + \cdots + \eta_n,$$

   where $\eta_1, \eta_2, \dots, \eta_n$ are independent and identically distributed *Rademacher* random variables:

$$\mathbb{P}(\eta_i = 1) = \mathbb{P}(\eta_i = -1) = \frac{1}{2}.$$

   Let $\mathcal{F}_n = \sigma(\eta_1, \dots, \eta_n) = \sigma(S_1, \dots, S_n)$, thus $(\mathcal{F}_n)$ is the filration generated by (S_n).

   (a) Find the discrete Doob-Meyer decomposition of (S_n^2), i.e., write $S_n^2 = A_n + M_n$, where (A_n) is predictable and (M_n) is a martingale. Find the simplest possible form of A_n.

   (b) Write M_n as the discrete stochastic integral $(H \cdot S)_n$ of a predictable process (H_n) with respect to the martingale (S_n). Find the simplest possible form of H_n.

5. You walk into a casino with one dollar in your pocket. The dealer tosses a fair coin in each round.
   Your betting strategy: you place one dollar on „heads" in each round. You play this game until you go bankrupt or you reach y dollars.

   (a) What is your chance of winning?

   (b) What is the expected number of rounds that you play until the game ends?

6. Assume given a random variable U with $\mathcal{N}(0,\sigma^2)$ distribution. We want to split U as the sum of $U^{(1)}+U^{(2)}=U$, where $U^{(1)}$ and $U^{(2)}$ are i.i.d. with $\mathcal{N}(0,\frac{1}{2}\sigma^2)$ distribution. How to find $U^{(1)}$ and $U^{(2)}$ given U?

   Here is the recipe: Let $Y\sim\mathcal{N}(0,1)$ be independent from U. Let $a\in\mathbb{R}_+$. Let
$$U^{(1)}=\frac{U}{2}+aY, \qquad U^{(2)}=\frac{U}{2}-aY.$$
   The question is how to choose the value of a is we want $U^{(1)}$ and $U^{(2)}$ to be i.i.d. with $\mathcal{N}(0,\frac{1}{2}\sigma^2)$ distribution?

   Hint: If two random variables have multivariate normal distribution then it is easy to characterize their independence using their covariance.

7. *Equivalent definitions of Brownian motion.* Show that if (B_t) is a stochastic process with an almost surely continuous trajectory then the following characterizations are equivalent:

   (a) $B_0=0$ and for any $0\le t_1<t_2<\cdots<t_n$ the increments $B_{t_i}-B_{t_{i-1}}$, $1\le i\le n$ are independent with normal distribution $B_{t_i}-B_{t_{i-1}}\sim\mathcal{N}(\mu,\sigma^2)$ with $\mu=0$ and $\sigma^2=t_i-t_{i-1}$.

   (b) (B_t) is a Gaussian process with $\mathbb{E}(B_t)=0$ and $\mathbf{Cov}(B_s,B_t)=s\wedge t$.

   Hint: You will have to show that a process that satisfies (a) also satisfies (b), and conversely, you will have to show that a process that satisfies (b) also satisfies (a). For the proof of the latter implication, keep in mind that a Gaussian process is uniquely determined by its mean function μ_t and auto-covariance function $\gamma(s,t)$.

8. Let (B_t) denote the standard Brownian motion.

   Denote by $\mathcal{F}_t=\sigma(B_s,\,0\le s\le t)$ the sigma-field that contains all events that can be determined by observing our Brownian motion up to time t. We call $(\mathcal{F}_t)_{t\ge0}$ the natural filtration generated by the process.

   We call a continuous-time stochastic process $(M_t)_{t\ge0}$ a martingale with respect to the filtration $(\mathcal{F}_t)$ if it is an adapted process which also satisfies $\mathbb{E}(X_t\,|\,\mathcal{F}_s)=X_s$ for any $0\le s\le t$.

   (a) Show that (B_t) is a martingale.

   (b) Show that (B_t^2-t) is a martingale.

   (c) Let $\lambda\in\mathbb{R}$ and $M_t=e^{\lambda B_t-t\lambda^2/2}$. Show that (M_t) is a martingale.

9. Use the reflection principle to show that $\max_{0\le s\le t}B_s$ has the same distribution as $|B_t|$.

10. *Lévy distribution:* Find the distribution of the hitting time T_x of level x, i.e., $T_x=\min\{\,t\,:\,B_t=x\,\}$. What is $\mathbb{E}(T_x)$?

11. Let $M_1:=\max_{0\le s\le1}B_s$. Show that M_1 has the same distribution as M_1-B_1.

12. Stationary Ornstein-Uhlenbeck process:

    (B_t) is standard Brownian motion. Let $\beta\in\mathbb{R}_+$ and define
$$X_t=e^{-\beta t}B(e^{2\beta t}), \qquad -\infty<t<+\infty.$$
    (a) Argue briefly that (X_t) is a Gaussian process.

    (b) Calculate $\mu_t=\mathbb{E}(X_t)$ and $\gamma_{s,t}=\mathrm{Cov}(X_s,X_t)$.

    (c) Show that (X_t) is a *stationary process*, i.e., show that for every $u\in\mathbb{R}$ the random vector
$$(X_{t_1+u},\ldots,X_{t_n+u})$$
    has the same distribution as $(X_{t_1},\ldots,X_{t_n})$. In words: the joint distributions are invariant under time shifts.
    Hint: More generally, show that a Gaussian process is stationary if and only if
$$\mu_t=\mu_{t+u}, \qquad \gamma_{s+u,t+u}=\gamma_{s,t} \qquad \text{for any} \quad u\in\mathbb{R}.$$
    Note that the first condition can be rephrased like this: μ_t is constant.
    Note that the second condition can be rephrased like this: there exists a function $f:\mathbb{R}_+\to\mathbb{R}$ such that $\gamma_{s,t}=f(|s-t|)$.

13. Use polarization to show that $\mathrm{Cov}(\int_0^t X_s \, \mathrm{d}B_s, \int_0^t Y_s \, \mathrm{d}B_s) = \mathbb{E}\left(\int_0^t X_s Y_s \, \mathrm{d}s\right)$.

14. Show that if $L_n = \sum_{k=1}^n B(t_{k-1}) \cdot (B(t_k) - B(t_{k-1}))$ and $\mathcal{I} = \int_0^t B_s \, \mathrm{d}B_s$ then

$$\mathbb{E}((\mathcal{I} - L_n)^2) = \sum_{k=1}^n \frac{1}{2}(t_k - t_{k-1})^2.$$

15. What is the distribution of $\int_0^1 s \, \mathrm{d}B_s$?

16. Let (B_t) denote standard Brownian motion. Show that (M_t) is a martingale, where

$$M_t = B_t^3 - 3tB_t$$

Hint: Use $B_t = B_s + (B_t - B_s)$, and also that $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

17. Let $(B(t))$ denote the standard Brownian motion.
    (a) Find the variance of $\int_0^2 B^2(s)\mathrm{d}B(s)$.
    (b) Find the covariance of $\int_0^1 e^{B(s)}\mathrm{d}B(s)$ and $\int_0^2 e^{-3B(s)}\mathrm{d}B(s)$.

18. Denote by (X_t) a left-continuous stochastic process adapted to $(\mathcal{F}_t)$ which is in $L_2(\Omega \times [0,2])$.
    Denote by $Y_t = \int_0^t X_u \, \mathrm{d}B_u$. Note that $Y_2 - Y_1 = \int_1^2 X_u \, \mathrm{d}B_u$.
    Calculate $\mathrm{Cov}(Y_1, Y_2 - Y_1)$.

19. Calculate $\mathbb{E}\left[\left(\int_1^2 B_s \, \mathrm{d}B_s\right)^2 \mid \mathcal{F}_1\right]$.

20. Let $Y_t = \int_0^t B_u \, \mathrm{d}u$. Calculate $\mathbb{E}(Y_t|\mathcal{F}_s)$, where $(\mathcal{F}_t)$ denotes the natural filtration of (B_t). Is (Y_t) a martingale?

21. Let $\lambda \in \mathbb{R}$. Use the differential form of Itô's formula to calculate the stochastic differentials

$$\mathrm{d}\cos(\lambda B_t) \qquad \text{and} \qquad \mathrm{d}\sin(\lambda B_t).$$

Now let us define the process (X_t) by

$$X_t := e^{i\lambda B_t} = \cos(\lambda B_t) + i\sin(\lambda B_t).$$

Sow that

$$\mathrm{d}X_t = i\lambda X_t \mathrm{d}B_t - \frac{1}{2}\lambda^2 X_t \mathrm{d}t$$

by calculating the stochastic differential of the real and imaginary part of (X_t) separately.

22. We say that two stochastic processes $(X(t))_{t\geq 0}$ and $(Y(t))_{t\geq 0}$ have the same law if for every choice of $n \geq 1$ and $0 \leq t_1 < t_2 < \cdots < t_n$ the joint distributions of $(X(t_1), X(t_2), \ldots, X(t_n))$ and $(Y(t_1), Y(t_2), \ldots, Y(t_n))$ are the same. Denote by $(B(t))$ the standard Brownian motion. Let

$$X(t) = \int_0^t (t-u) \, \mathrm{d}B(u) \qquad\qquad Y(t) = \int_0^t B(u) \, \mathrm{d}u$$

Show that $(X(t))_{t\geq 0}$ and $(Y(t))_{t\geq 0}$ have the same law.

Hint: Both $(X(t))_{t\geq 0}$ and $(Y(t))_{t\geq 0}$ are Gaussian processes, so you only need to check that $\mathbb{E}[X(t)] = \mathbb{E}[Y(t)]$ for all $t \geq 0$ and $\mathrm{Cov}(X_s, X_t) = \mathrm{Cov}(Y_s, Y_t)$ for all $0 \leq s \leq t$.

You will need some facts about Itô integrals with a deterministic integrand.

Hint 2: By Fubini's theorem, expectations and integrals can be interchanged. Actually double integrals and expectations can also be interchanged:

$$\mathbb{E}\left[\int_a^b \int_c^d Z_{u,v} \, \mathrm{d}u \, \mathrm{d}v\right] = \int_a^b \int_c^d \mathbb{E}\left[Z_{u,v}\right] \, \mathrm{d}u \, \mathrm{d}v$$

This observation will be useful when you calculate the autocovariance function of $(Y(t))$.

23. Calculate $\mathrm{Cov}(X_s, X_t)$, where $X_t = \int_0^t B_u f'(u)\,\mathrm{d}u$.

24. Show that $\int_0^t B_u\,\mathrm{d}u = \int_0^t (t-u)\,\mathrm{d}B_u$.

25. Let $M_t = B_t^3 - 3tB_t$. Calculate the stochastic differential of M_t and show that (M_t) is a martingale.

26. Let $V_t = \int_0^t \exp(\beta(u-t))\,\mathrm{d}B_u$. Is (V_t) a martingale?

27. Let $f : \mathbb{R}_+ \to \mathbb{R}$ denote a deterministic continuous function. Let

$$M_t = \exp\left(i \int_0^t f(s)\,\mathrm{d}B_s + \frac{1}{2}\int_0^t f^2(s)\,\mathrm{d}s \right).$$

Show that (M_t) is a martingale and write it as a stochastic integral w.r.t. (B_s).

28. Let $X = \exp\left(i \int_0^t f(s)\,\mathrm{d}B_s \right)$. Find the adapted process $(\sigma(t))_{t=0}^T$ for which $X = \mathbb{E}(X) + \int_0^T \sigma(t)\,\mathrm{d}B_t$.

29. We have seen in class that if $Y_t = \int_0^t \sigma_s\,\mathrm{d}B_s$, then $M_t = Y_t^2 - [Y]_t$ is a martingale.

Use Itô's formula for Itô processes to show that $M_t = \int_0^t \widetilde{\sigma}_s\,\mathrm{d}B_s$ for some process $(\widetilde{\sigma}_s)$. Give an explicit formula for $\widetilde{\sigma}_s$.

30. The Itô process (X_t) has stochastic differential

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t$$

with drift coefficient $\mu(x) = cx$ (where $c > 0$) and diffusion coefficient $\sigma(x) = x^a$ (where $a > 0$).

Let us define

$$Y_t = X_t^b$$

for some $b \in \mathbb{R}$.

(a) Calculate the stochastic differential $\mathrm{d}Y_t$ using Itô's formula for Itô processes.

(b) How to choose b if we want the diffusion coefficient of (Y_t) to be constant?

31. We have already seen that $M_t = \exp(\lambda B_t - t\frac{\lambda^2}{2})$ is a martingale.

Your goal is to prove this again using stochastic calculus.

Please use the notation $M_t = X_t Y_t$, where $X_t = e^{\lambda B_t}$ and $Y_t = e^{-t\frac{\lambda^2}{2}}$.

32. Let us fix $T > 0$ and denote $\Phi(x) = \int_{-\infty}^x \varphi(y)\,\mathrm{d}y$, where $\varphi(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$.

Show that

$$M_t = \Phi\left(\frac{B_t}{\sqrt{T-t}} \right), \qquad 0 \le t < T$$

is a martingale.

33. Let $X = \mathbb{1}[B_T > 1]$. Calculate $\mathbb{E}(X \mid \mathcal{F}_t)$ and find the adapted process $(\sigma(t))_{t=0}^T$ for which

$$X = \mathbb{E}(X) + \int_0^T \sigma(t)\,\mathrm{d}B_t.$$

34. How to choose the differentiable function $f : \mathbb{R} \to \mathbb{R}$ so that $M_t = f(t)\cos(B_t)$ is a martingale with $M_0 = 1$? Use this martingale for something interesting.

35. If a function $f : \mathbb{R}^n \to \mathbb{R}$ satisfies $\Delta f \equiv 0$, then we say that f is a harmonic function.

It is a fact from complex analysis that the real part of complex analytic function is a harmonic function.

(a) Let $g : \mathbb{C} \to \mathbb{C}$ be defined by $g(z) = z^3$ (thus g is a complex analytic function). Let us define

$$f(x, y) = \mathrm{Re}(g(x + iy)).$$

Write an explicit formula for $f(x, y)$ and verify that in this case we indeed have $\Delta f = f_{xx} + f_{yy} \equiv 0$.

(b) Use (a) to show that if $B_1(t)$ and $B_2(t)$ are independent Brownian motions, then $B_1^3(t) - 3B_1(t)B_2^2(t)$ is a martingale.

36. Let $\underline{B}_t = (B_1(t), \ldots, B_d(t))$ denote d-dimensional Brownian motion started from $\underline{B}_0 = \underline{x}_0 \in \mathbb{R}^d$. Let $\|\underline{x}\| = \sqrt{x_1^2 + \cdots + x_d^2}$ denote the Euclidean norm. Let $f(\underline{x}) = \|\underline{x}\|^2$. Let $R \geq \|\underline{x}_0\|$ and denote by

$$\tau = \min\{\, t \; : \; \|\underline{B}_t\| = R \,\}$$

the exit time from a ball of radius R.

(a) Calculate the stochastic differential $\mathrm{d}f(\underline{B}_t)$.

(b) Show that $\|\underline{B}_t\|^2 - d \cdot t$ is a martingale.

(c) Use the optional stopping theorem to calculate $\mathbb{E}(\tau)$.
Instruction: You don't have to check that the optional stopping theorem can be applied here.

37. Solve the Langevin equation, i.e., find an Ito process (X_t) such that $\mathrm{d}X_t = -\alpha X_t \, \mathrm{d}t + \sigma \mathrm{d}B_t$ and $X_0 = x_0$ (where $\alpha, \sigma \in \mathbb{R}_+$ and $x_0 \in \mathbb{R}$).

38. *Stochastic exponential:* given an Ito process (X_t), find the Ito process (U_t) for which $U_t = 1 + \int_0^t U_s \, \mathrm{d}X_s$ holds for all $t \geq 0$.

39. *Geometric Brownian motion:* solve the SDE $\mathrm{d}S_t = rS_t \mathrm{d}t + \sigma S_t \mathrm{d}B_t$ with initial condition $S_0 = s_0$, where $r, \sigma, s_0 \in \mathbb{R}_+$

40. Solve the SDE $\mathrm{d}X_t = \frac{b - X_t}{T - t} \, \mathrm{d}t + \mathrm{d}B_t$ with $X_0 = a$ on the interval $t \in [0, T)$, where $a, b \in \mathbb{R}$ and $T \in \mathbb{R}_+$.

41. Let (X_t) solve the SDE $\mathrm{d}X_t = \frac{1}{2} X_t \, \mathrm{d}t + \mathrm{d}B_t$ with $X_0 = x_0 \in \mathbb{R}$. Let $T_x = \inf\{\, t \; : \; X_t = x \}$. Let $a < x_0 < b$. Find $\mathbb{P}(T_a < T_b)$. Calculate $\mathbb{P}(T_{+\infty} < T_{-\infty})$ using the strong solution of the SDE.

42. Let $\underline{B}_t = (B_1(t), B_2(t), B_3(t))$ denote 3-dimensional Brownian motion started from $\underline{B}_0 = \underline{x}_0 \in \mathbb{R}^d$, $\underline{x}_0 \neq \underline{0}$. Let $\|\underline{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ denote the Euclidean norm. Let $f(\underline{x}) = 1/\|\underline{x}\|$. Calculate the stochastic differential $\mathrm{d}f(\underline{B}_t)$ and show that the drift term vanishes.

43. Let $\underline{B}_t = (B_1(t), B_2(t), B_3(t))$ denote 3-dimensional Brownian motion started from $\underline{B}_0 = \underline{x}_0 \in \mathbb{R}^d$, $\underline{x}_0 \neq \underline{0}$. Let $T_r = \inf\{\, t : \|\underline{B}_t\| = r \}$. Let $0 < a < \|\underline{x}_0\| < b$.

(a) Find $\mathbb{P}(T_a < T_b)$.

(b) Show that $(\underline{B}_t)$ never hits the origin.

44. *Stochastic harmonic oscillator:* let us consider the following system of SDE's:

$$\mathrm{d}X_t = -Y_t \, \mathrm{d}B_t, \qquad \mathrm{d}Y_t = X_t \, \mathrm{d}B_t, \qquad X_0 = 1, \qquad Y_0 = 0.$$

Let $Z_t = X_t^2 + Y_t^2$. Find Z_t.

45. *Bessel process:* Let $(\underline{B}_t)$ denote d-dimensional Brownian motion. Show that if $Y_t = \|\underline{B}_t\|$, then (Y_t) is a weak solution of the SDE

$$\mathrm{d}Y_t = \frac{d-1}{2} \frac{1}{Y_t} + \mathrm{d}\widetilde{B}_t.$$

46. *Stochastic logarithm:* given an Ito process (U_t) and X_0, find the Ito process (X_t) for which

$$U_t = 1 + \int_0^t U_s \, \mathrm{d}X_s, \qquad t \geq 0.$$

47. Argue that geometric Brownian motion is a time-homogeneous Markov process and find its transition probability density function.

48. Solve $\mathrm{d}X_t = X_t^3 \mathrm{d}t + X_t^2 \mathrm{d}B_t$ with $X_0 = 1$ and argue why existence of solutions for all $t \in \mathbb{R}_+$ fails.
Hint: The solution is of form $X_t = f(B_t)$ for some deterministic function f.

49. Solve $\mathrm{d}X_t = 3X_t^{1/3}\mathrm{d}t3 + X_t^{2/3}\mathrm{d}B_t$ with $X_0 = 0$ and argue why uniqueness of solutions fails.

    Hint: The solution is of form $X_t = f(B_t)$ for some deterministic function f.

50. *General linear SDE:* Given the Ito processes (X_t), (Y_t), find the Ito process (V_t) for which $\mathrm{d}V_t = V_t\mathrm{d}X_t + \mathrm{d}Y_t$ and $V_0 = v_0$.

51. *Squared Bessel process:* given an Ito process that solves $\mathrm{d}Y_t = \frac{n-1}{2}\frac{1}{Y_t} + \mathrm{d}B_t$, let $X_t = Y_t^2$. Show that $\mathrm{d}X_t = n\mathrm{d}t + 2\sqrt{X_t}\mathrm{d}B_t$.

52. *Hitting probabilities for Bessel process:* Let $0 < d \neq 2$, let $\mathrm{d}Y_t = \frac{d-1}{2}\frac{1}{Y_t} + \mathrm{d}B_t$, $Y_0 = y_0$. Let $0 < a < y_0 < b$. Let $T_y = \inf\{\, t : Y_t = y\}$. Find $\mathbb{P}(T_a < T_b)$.

53. Find a 2-dimensional SDE such that the corresponding infinitesimal generator is

$$Af = f_{xx} + 3f_{xy} + \frac{5}{2}f_{yy} + f_x - f_y$$

54. If $X \sim \mathcal{N}(0,1)$ under $\mathbb{P}$ and $u \in \mathbb{R}$, find $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = m(X)$ such that $u + X \sim \mathcal{N}(0,1)$ under $\mathbb{Q}$

55. Find the joint probability density function of $\max_{0\le t\le 1} \widetilde{B}_t$ and $\widetilde{B}_1$, where $(\widetilde{B}_t)$ is BM with constant drift.