Exam themes, preparation list, Stochastic Analysis, 2023 spring

The following list contains the topics covered in class. On the top of this list, remember that the homework exercises and their solutions are also part of the exam material. Exam questions may pertain to exercises, definitions, theorems, proofs, anything really. Note that the numbering of the lectures/sections below matches with the numbering of lecture note pdf files in the Class Materials folder in Microsoft Teams, but it does not perfectly match the order and the chronology in witch I presented these topics in 2023.

1. Lecture

- Measure-theoretic probability, probability space, probability measure, simple random variable
- expectation, e.g., of simple random variables
- sigma-algebras generated by random variables
- independence
- Show that if X and Y are both simple random variables, moreover X and Y are independent, then

$$\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y).$$

- conditional expectation: discrete case, absolutely continuous case, general case
- properties of conditional expectaton

2. Lecture

- Properties of conditional expectation: trivial sigma-algebra, measurable random variable, linearity, separation of known info, tower property, independent info can be ignored, conditional Jensen
- Filtration, adapted process, predictable process, natural filtration of process
- martingale, submartingale, supermartingale
- discrete stochastic integral of a predictable process with respect to a martingale is a martingale

3. Lecture (we skipped some of this in 2023)

- Proof of properties of conditional expectation: separation of known info, tower property, independent info can be ignored, conditional Jensen
- Let (X, Y) denote a pair of simple random variables defined on the same probability space. Denote by $\{x_1, \dots, x_n\}$ the set of possible values of X and denote by $\{y_1, \dots, y_m\}$ the set of possible values of Y. Let $p_{i,j} = \mathbb{P}(X = x_i, Y = y_j)$ and $p_i = \sum_{j=1}^m p_{i,j}$. Denote by

$$H(x_i) := \sum_{j=1}^{m} y_j \frac{p_{i,j}}{p_i}, \quad i = 1, \dots, n.$$

  1. Let $A_i = \{X = x_i\}$, $i = 1, \dots, n$. Show that the sigma-algebra $\sigma(X)$ generated by X consists of the events of form $A_I = \cup_{i \in I} A_i$, where $I \subseteq \{1, \dots, n\}$.
  2. Prove that the random variable $H(X)$ satisfies the abstract definition of $\mathbb{E}(\, Y \mid \sigma(X)\,)$.
     Hint: You have to check that $Z := H(X)$ satisfies $\mathbb{E}(|H(X)|) < +\infty$, $H(X)$ is $\sigma(X)$-measurable and that $\mathbb{E}(Y \mathbb{1}_A) = \mathbb{E}(H(X)\mathbb{1}_A)$ for any $A \in \sigma(X)$.

- Conditional expectation is the best guess if we want to minimize mean squared error

4. Lecture

- Let (S_n) denote a 1-dimensional simple symmetric random walk. That is:

  $$S_n = \eta_1 + \eta_2 + \cdots + \eta_n,$$

  where $\eta_1, \eta_2, \ldots, \eta_n$ are independent and identically distributed *Rademacher* random variables:

  $$\mathbb{P}(\eta_i = 1) = \mathbb{P}(\eta_i = -1) = \frac{1}{2}.$$

  Let $\mathcal{F}_n = \sigma(\eta_1, \ldots, \eta_n) = \sigma(S_1, \ldots, S_n)$, thus $(\mathcal{F}_n)$ is the filtration generated by (S_n).

  1. Show that (S_n) is a martingale. (You will have to use some of the properties listed in the lecture notes – name them when you use them)
  2. Show that if $\lambda \in \mathbb{R}$ and

     $$M_n = e^{\lambda S_n - n \ln(\cosh(\lambda))},$$

     then (M_n) is a martingale. (Again, name the properties that you use)
     Hint: What is the moment generating function of a Rademacher random variable?

- Gambling: meaning of discrete stochastic integral (net profit of betting strategy). „You can't beat the system"
- Convexity and submartingales
- Stopping time, optional stopping theorem
- Discrete Doob-Meyer decomposition

5. Lecture

- Discrete Doob-Meyer decomposition of submartingales
- Let (S_n) denote a 1-dimensional simple symmetric random walk. That is:

  $$S_n = \eta_1 + \eta_2 + \cdots + \eta_n,$$

  where $\eta_1, \eta_2, \ldots, \eta_n$ are independent and identically distributed *Rademacher* random variables:

  $$\mathbb{P}(\eta_i = 1) = \mathbb{P}(\eta_i = -1) = \frac{1}{2}.$$

  Let $\mathcal{F}_n = \sigma(\eta_1, \ldots, \eta_n) = \sigma(S_1, \ldots, S_n)$, thus $(\mathcal{F}_n)$ is the filration generated by (S_n).

  1. Find the discrete Doob-Meyer decomposition of (S_n^2), i.e., write $S_n^2 = A_n + M_n$, where (A_n) is predictable and (M_n) is a martingale. Find the simplest possible form of A_n.
  2. Write M_n as the discrete stochastic integral $(H \cdot S)_n$ of a predictable process (H_n) with respect to the martingale (S_n). Find the simplest possible form of H_n.

- You walk into a casino with one dollar in your pocket. The dealer tosses a fair coin in each round.

  Your betting strategy: you place one dollar on „heads" in each round. You play this game until you go bankrupt or you reach y dollars.

  1. What is your chance of winning?
  2. What is the expected number of rounds that you play until the game ends?

- Multivariate normal distribution and its properties

6. Lecture

- Paul Lévy's construction of Brownian motion
- Assume given a random variable U with $\mathcal{N}(0,\sigma^2)$ distribution. We want to split U as the sum of $U^{(1)} + U^{(2)} = U$, where $U^{(1)}$ and $U^{(2)}$ are i.i.d. with $\mathcal{N}(0, \frac{1}{2}\sigma^2)$ distribution. How to find $U^{(1)}$ and $U^{(2)}$ given U?

  Here is the recipe: Let $Y \sim \mathcal{N}(0,1)$ be independent from U. Let $a \in \mathbb{R}_+$. Let

  $$U^{(1)} = \frac{U}{2} + aY, \qquad U^{(2)} = \frac{U}{2} - aY.$$

  The question is how to choose the value of a is we want $U^{(1)}$ and $U^{(2)}$ to be i.i.d. with $\mathcal{N}(0, \frac{1}{2}\sigma^2)$ distribution?

  Hint: If two random variables have multivariate normal distribution then it is easy to characterize their independence using their covariance.

7. Lecture

- Solution of HW1.4 without integrals
- Continuous-time stochastic processes (filtration, stopping time, adapted process, martingales)
- Doob's martingale construction
- Gaussian processes
- Brownian motion
- BM and random walk
- *Equivalent definitions of Brownian motion.* Show that if (B_t) is a stochastic process with an almost surely continuous trajectory then the following characterizations are equivalent:

  1. $B_0 = 0$ and for any $0 \le t_1 < t_2 < \cdots < t_n$ the increments $B_{t_i} - B_{t_{i-1}}$, $1 \le i \le n$ are independent with normal distribution $B_{t_i} - B_{t_{i-1}} \sim \mathcal{N}(\mu, \sigma^2)$ with $\mu = 0$ and $\sigma^2 = t_i - t_{i-1}$.
  2. (B_t) is a Gaussian process with $\mathbb{E}(B_t) = 0$ and $\mathbf{Cov}(B_s, B_t) = s \wedge t$.

  Hint: You will have to show that a process that satisfies (a) also satisfies (b), and conversely, you will have to show that a process that satisfies (b) also satisfies (a). For the proof of the latter implication, keep in mind that a Gaussian process is uniquely determined by its mean function μ_t and auto-covariance function $\gamma(s,t)$.

- Scale invariance of BM
- Let (B_t) denote the standard Brownian motion.

  Denote by $\mathcal{F}_t = \sigma(B_s, 0 \le s \le t)$ the sigma-field that contains all events that can be determined by observing our Brownian motion up to time t. We call $(\mathcal{F}_t)_{t \ge 0}$ the natural filtration generated by the process.

  We call a continuous-time stochastic process $(M_t)_{t\ge 0}$ a martingale with respect to the filtration $(\mathcal{F}_t)$ if it is an adapted process which also satisfies $\mathbb{E}(X_t \,|\, \mathcal{F}_s) = X_s$ for any $0 \le s \le t$.

  (a) Show that (B_t) is a martingale.
  (b) Show that $(B_t^2 - t)$ is a martingale.
  (c) Let $\lambda \in \mathbb{R}$ and $M_t = e^{\lambda B_t - t\lambda^2/2}$. Show that (M_t) is a martingale.

8. Lecture

- Total variation
- Stieltjes integral def
- Quadratic variation
- Mutual variation, bilinearity, polarization
- Relationship of total variation and quadratic variation
- Quadratic variation of BM
- Mutual variation of two independent BMs
- Markov property for continuous-time processes

9. Lecture

- Proof of the theorem that Stieltjes integral is well-defined
- BM is nowhere differentiable (there are no exceptional times) (with proof)
- Local maxima of BM (there are exceptional times)

10. Lecture

- Minus BM is a BM
- Time reversal of BM is BM
- Transition p.d.f. of a time-homogeneous Markov process, Chapman-Kolmogorov equations
- BM is a time-homogeneous Markov process. Identification of its transition p.d.f.
- Strong Markov property
- BM has the strong Markov property
- Use the reflection principle to show that $\max_{0\le s\le t} B_s$ has the same distribution as $|B_t|$.
- Lévy distribution: Find the distribution of the hitting time T_x of level x, i.e., $T_x = \min\{\, t \,:\, B_t = x\,\}$. What is $\mathbb{E}(T_x)$?
- Let $M_1 := \max_{0\le s\le 1} B_s$. Show that M_1 has the same distribution as $M_1 - B_1$.

11. Lecture

- Simple predictable process, integral of it w.r.t. BM, intuitive meaning (financial math)
- Properties of stoch. integral of simple pred. process w.r.t. BM: linearity, Itô isometry
- The $\|\cdot\|_*$ norm
- Extension of Itô integral using Itô isometry
- Stationary Ornstein-Uhlenbeck process:
  (B_t) is standard Brownian motion. Let $\beta \in \mathbb{R}_+$ and define

  $$X_t = e^{-\beta t} B(e^{2\beta t}), \qquad -\infty < t < +\infty.$$

  (a) Argue briefly that (X_t) is a Gaussian process.
  (b) Calculate $\mu_t = \mathbb{E}(X_t)$ and $\gamma_{s,t} = \mathrm{Cov}(X_s, X_t)$.

(c) Show that (X_t) is a *stationary process*, i.e., show that for every $u \in \mathbb{R}$ the random vector

$$(X_{t_1+u}, \ldots, X_{t_n+u})$$

has the same distribution as $(X_{t_1}, \ldots, X_{t_n})$. In words: the joint distributions are invariant under time shifts.

Hint: More generally, show that a Gaussian process is stationary if and only if

$$\mu_t = \mu_{t+u}, \qquad \gamma_{s+u,t+u} = \gamma_{s,t} \qquad \text{for any} \quad u \in \mathbb{R}.$$

Note that the first condition can be rephrased like this: μ_t is constant.

Note that the second condition can be rephrased like this: there exists a function $f : \mathbb{R}_+ \to \mathbb{R}$ such that $\gamma_{s,t} = f(|s-t|)$.

12. Lecture

- More about the O-U process
- Relationship between Itô and Stieltjes integrals
- Properties of Itô integral: linearity, variance of integral
- Use polarization to show that $\mathrm{Cov}(\int_0^t X_s \, \mathrm{d}B_s, \int_0^t Y_s \, \mathrm{d}B_s) = \mathbb{E}\left(\int_0^t X_s Y_s \, \mathrm{d}s\right)$.
- Itô integral is a continuous martingale (if the integrand is a simple predictable process) (Skipped this in 2023)
- Submartingale inequality (Skipped this in 2023)
- Itô integral is a continuous martingale (if the integrand is a „general" process) (Skipped this in 2023)

13. Lecture

- Integral of BM w.r.t. BM
- Show that if $L_n = \sum_{k=1}^n B(t_{k-1}) \cdot (B(t_k) - B(t_{k-1}))$ and $\mathcal{I} = \int_0^t B_s \, \mathrm{d}B_s$ then

$$\mathbb{E}((\mathcal{I} - L_n)^2) = \sum_{k=1}^n \frac{1}{2}(t_k - t_{k-1})^2.$$

- What is the distribution of $\int_0^1 s \, \mathrm{d}B_s$?
- integral of a deterministic function w.r.t. BM (Gaussian process), covariance structure
- Let (B_t) denote standard Brownian motion. Show that (M_t) is a martingale, where

$$M_t = B_t^3 - 3tB_t$$

  Hint: Use $B_t = B_s + (B_t - B_s)$, and also that $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ as well as the result of HW1.3(c) about the moments of normal random variables.
- New def of stationary O-U process as a stochastic integral
- Let $(B(t))$ denote the standard Brownian motion.
  (a) Find the variance of $\int_0^2 B^2(s) \mathrm{d}B(s)$.
  (b) Find the covariance of $\int_0^1 e^{B(s)} \mathrm{d}B(s)$ and $\int_0^2 e^{-3B(s)} \mathrm{d}B(s)$.

14. Lecture

- Conditional Itô isometry
- Doob-Meyer decomposition of the square of a stochastic integral w.r.t. BM
- Quadratic/mutual variation of Itô integrals w.r.t. BM
- Total variation of an Itô integral w.r.t. BM
- Definition of Itô process, unique decomposition
- Chain rule of regular calculus
- Statement of Itô formula for BM

15. Lecture

- Proof of Itô formula
- Denote by (X_t) a left-continuous stochastic process adapted to $(\mathcal{F}_t)$ which is in $L_2(\Omega \times [0,2])$.
  Denote by $Y_t = \int_0^t X_u \,\mathrm{d}B_u$. Note that $Y_2 - Y_1 = \int_1^2 X_u \,\mathrm{d}B_u$.
  Calculate $\mathrm{Cov}(Y_1, Y_2 - Y_1)$.
- Independence vs. zero correlation: the difference between the case of deterministic integrand and general adapted integrand
- Calculate $\mathbb{E}\left[\left(\int_1^2 B_s \,\mathrm{d}B_s\right)^2 \mid \mathcal{F}_1\right]$.

16. Lecture

- Itô calculus (differential form of Itô processes)
- Short heuristic proof of Itô's formula using stochastic differentials
- Infinitesimal drift, infinitesimal variance
- Mutual/quadratic variation of Itô processes
- Equivalent definitions of an Itô process with finite total variation
- Let $Y_t = \int_0^t B_u \,\mathrm{d}u$. Calculate $\mathbb{E}(Y_t|\mathcal{F}_s)$, where $(\mathcal{F}_t)$ denotes the natural filtration of (B_t). Is (Y_t) a martingale?

17. Lecture

- Itô integral of the absolute value is not bigger than the absolute value of the Itô integral
- Conditional Fubini proof
- two equivalent definitions of Brownian bridge
- Let $\lambda \in \mathbb{R}$. Use the differential form of Itô's formula to calculate the stochastic differentials

  $$\mathrm{d}\cos(\lambda B_t) \qquad \text{and} \qquad \mathrm{d}\sin(\lambda B_t).$$

  Now let us define the process (X_t) by

  $$X_t := e^{i\lambda B_t} = \cos(\lambda B_t) + i\sin(\lambda B_t).$$

  Sow that

  $$\mathrm{d}X_t = i\lambda X_t \mathrm{d}B_t - \frac{1}{2}\lambda^2 X_t \mathrm{d}t$$

  by calculating the stochastic differential of the real and imaginary part of (X_t) separately.

- Stochastic integral w.r.t. an Itô process
- Itô's formula for Itô processes (with heuristic proof)

18. Lecture

- We say that two stochastic processes $(X(t))_{t\geq 0}$ and $(Y(t))_{t\geq 0}$ have the same law if for every choice of $n \geq 1$ and $0 \leq t_1 < t_2 < \cdots < t_n$ the joint distributions of $(X(t_1), X(t_2), \ldots, X(t_n))$ and $(Y(t_1), Y(t_2), \ldots, Y(t_n))$ are the same. Denote by $(B(t))$ the standard Brownian motion. Let

  $$X(t) = \int_0^t (t-u)\,\mathrm{d}B(u) \qquad\qquad Y(t) = \int_0^t B(u)\,\mathrm{d}u$$

  Show that $(X(t))_{t\geq 0}$ and $(Y(t))_{t\geq 0}$ have the same law.

  Hint: Both $(X(t))_{t\geq 0}$ and $(Y(t))_{t\geq 0}$ are Gaussian processes, so you only need to check that $\mathbb{E}[X(t)] = \mathbb{E}[Y(t)]$ for all $t \geq 0$ and $\mathrm{Cov}(X_s, X_t) = \mathrm{Cov}(Y_s, Y_t)$ for all $0 \leq s \leq t$.

  You will need some facts about Itô integrals with a deterministic integrand.

  Hint 2: By Fubini's theorem, expectations and integrals can be interchanged. Actually double integrals and expectations can also be interchanged:

  $$\mathbb{E}\left[\int_a^b \int_c^d Z_{u,v}\,\mathrm{d}u\,\mathrm{d}v\right] = \int_a^b \int_c^d \mathbb{E}\left[Z_{u,v}\right]\,\mathrm{d}u\,\mathrm{d}v$$

  This observation will be useful when you calculate the autocovariance function of $(Y(t))$.
- Product rule of classical calculus
- Stochastic product rule (with heuristic proof)
- Calculate $\mathrm{Cov}(X_s, X_t)$, where $X_t = \int_0^t B_u f'(u)\,\mathrm{d}u$.
- Show that $\int_0^t B_u\,\mathrm{d}u = \int_0^t (t-u)\,\mathrm{d}B_u$.
- Let $M_t = B_t^3 - 3tB_t$. Calculate the stochastic differential of M_t and show that (M_t) is a martingale.
- $Z_t = \int_0^t X_s\,\mathrm{d}Y_s$ is not necessarily a martingale.
- Let $V_t = \int_0^t \exp\left(\beta(u-t)\right)\,\mathrm{d}B_u$. Is (V_t) a martingale?

19. Lecture

- Martingale representation theorem (with proof)
- Let $f : \mathbb{R}_+ \to \mathbb{R}$ denote a deterministic continuous function. Let

  $$M_t = \exp\left(i\int_0^t f(s)\,\mathrm{d}B_s + \frac{1}{2}\int_0^t f^2(s)\,\mathrm{d}s\right).$$

  Show that (M_t) is a martingale and write it as a stochastic integral w.r.t. (B_s).
- Let $X = \exp\left(i\int_0^t f(s)\,\mathrm{d}B_s\right)$. Find the adapted process $(\sigma(t))_{t=0}^T$ for which $X = \mathbb{E}(X) + \int_0^T \sigma(t)\,\mathrm{d}B_t$.

20. Lecture

- We have seen in class that if $Y_t = \int_0^t \sigma_s \, \mathrm{d}B_s$, then $M_t = Y_t^2 - [Y]_t$ is a martingale (see lecture 14).

  Use Itô's formula for Itô processes to show that $M_t = \int_0^t \widetilde{\sigma}_s \, \mathrm{d}B_s$ for some process $(\widetilde{\sigma}_s)$. Give an explicit formula for $\widetilde{\sigma}_s$.

- The Itô process (X_t) has stochastic differential

  $$\mathrm{d}X_t = \mu(X_t)\, \mathrm{d}t + \sigma(X_t)\, \mathrm{d}B_t$$

  with drift coefficient $\mu(x) = cx$ (where $c > 0$) and diffusion coefficient $\sigma(x) = x^a$ (where $a > 0$). Let us define

  $$Y_t = X_t^b$$

  for some $b \in \mathbb{R}$.

  (a) Calculate the stochastic differential $\mathrm{d}Y_t$ using Itô's formula for Itô processes.

  (b) How to choose b if we want the diffusion coefficient of (Y_t) to be constant?

- We have already seen that $M_t = \exp(\lambda B_t - t\frac{\lambda^2}{2})$ is a martingale (see lecture 7).

  Your goal is to prove this again using stochastic calculus. (Skipped this in 2023)

  Please use the notation $M_t = X_t Y_t$, where $X_t = e^{\lambda B_t}$ and $Y_t = e^{-t\frac{\lambda^2}{2}}$.

- Time-dependent Itô formula (with heuristic proof)

- Let us fix $T > 0$ and denote $\Phi(x) = \int_{-\infty}^x \varphi(y)\, \mathrm{d}y$, where $\varphi(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$.

  Show that

  $$M_t = \Phi\left(\frac{B_t}{\sqrt{T-t}}\right), \qquad 0 \le t < T$$

  is a martingale.

- Let $X = \mathbb{1}[B_T > 1]$. Calculate $\mathbb{E}(X \mid \mathcal{F}_t)$ and find the adapted process $(\sigma(t))_{t=0}^T$ for which

  $$X = \mathbb{E}(X) + \int_0^T \sigma(t)\, \mathrm{d}B_t.$$

21. Lecture

- How to choose the differentiable function $f : \mathbb{R} \to \mathbb{R}$ so that $M_t = f(t)\cos(B_t)$ is a martingale with $M_0 = 1$? Use this martingale for something interesting.
- Itô process driven by d-dimensional BM, multi-dimensional Itô process
- Mutual/quadratic variation of Itô processes driven by the same d-dimensional BM
- Integration w.r.t. an Itô process driven by d-dimensional BM
- Itô's formula (several variables)
- A remark about the optional stopping theorem (OST) (homework 6.2)
- OST cannot be applied for Brownian motion and the hitting time of level 1.

22. Lecture

- Taylor formula (several variables) (Skipped in 2023)
- Heuristic proof of multi-variable Itô formula (Skipped in 2023)
- Paul Lévy's characterization of BM (with proof)
- Crash course on characteristic functions

23. Lecture

- Multi-variable Itô formula for d-dimensional BM (Laplace operator appears)
- Harmonic functions and martingales
- If a function $f : \mathbb{R}^n \to \mathbb{R}$ satisfies $\Delta f \equiv 0$, then we say that f is a harmonic function.
  It is a fact from complex analysis that the real part of complex analytic function is a harmonic function.
  (a) Let $g : \mathbb{C} \to \mathbb{C}$ be defined by $g(z) = z^3$ (thus g is a complex analytic function). Let us define
  $$f(x, y) = \mathrm{Re}(g(x + iy)).$$
  Write an explicit formula for $f(x, y)$ and verify that in this case we indeed have $\Delta f = f_{xx} + f_{yy} \equiv 0$.
  (b) Use (a) to show that if $B_1(t)$ and $B_2(t)$ are independent Brownian motions, then $B_1^3(t) - 3B_1(t)B_2^2(t)$ is a martingale.
- All Itô formulas (and the product rule) are special cases of the multi-variable Itô formula
- Let $\underline{B}_t = (B_1(t), \dots, B_d(t))$ denote d-dimensional Brownian motion started from $\underline{B}_0 = \underline{x}_0 \in \mathbb{R}^d$.
  Let $\|\underline{x}\| = \sqrt{x_1^2 + \cdots + x_d^2}$ denote the Euclidean norm. Let $f(\underline{x}) = \|\underline{x}\|^2$.
  Let $R \geq \|\underline{x}_0\|$ and denote by
  $$\tau = \min\{\, t \,:\, \|\underline{B}_t\| = R \,\}$$
  the exit time from a ball of radius R.
  (a) Calculate the stochastic differential $\mathrm{d}f(\underline{B}_t)$.
  (b) Show that $\|\underline{B}_t\|^2 - d \cdot t$ is a martingale.
  (c) Use the optional stopping theorem to calculate $\mathbb{E}(\tau)$.
  Instruction: You don't have to check that the optional stopping theorem can be applied here.
- Solve the Langevin equation, i.e., find an Ito process (X_t) such that $\mathrm{d}X_t = -\alpha X_t \,\mathrm{d}t + \sigma \mathrm{d}B_t$ and $X_0 = x_0$ (where $\alpha, \sigma \in \mathbb{R}_+$ and $x_0 \in \mathbb{R}$). (The solution turns out to be an O-U process)

24. Lecture

- Exponential of classical calculus vs stochastic exponential
- Stochastic exponential: given an Ito process (X_t), find the Ito process (U_t) for which $U_t = 1 + \int_0^t U_s \,\mathrm{d}X_s$ holds for all $t \geq 0$.
- Geometric Brownian motion: solve the SDE $\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t \mathrm{d}B_t$ with initial condition $S_0 = s_0$, where $r, \sigma, s_0 \in \mathbb{R}_+$
- Brownian bridge: solve the SDE $\mathrm{d}X_t = \frac{b - X_t}{T - t}\,\mathrm{d}t + \mathrm{d}B_t$ with $X_0 = a$ on the interval $t \in [0, T)$, where $a, b \in \mathbb{R}$ and $T \in \mathbb{R}_+$.
- Let (X_t) solve the SDE $\mathrm{d}X_t = \frac{1}{2}X_t \,\mathrm{d}t + \mathrm{d}B_t$ with $X_0 = x_0 \in \mathbb{R}$. Let $T_x = \inf\{\, t \,:\, X_t = x\}$. Let $a < x_0 < b$. Find $\mathbb{P}(T_a < T_b)$. Calculate $\mathbb{P}(T_{+\infty} < T_{-\infty})$ using the strong solution of the SDE.

25. Lecture

- Let $\underline{B}_t = (B_1(t), B_2(t), B_3(t))$ denote 3-dimensional Brownian motion started from $\underline{B}_0 = \underline{x}_0 \in \mathbb{R}^d$, $\underline{x}_0 \neq \underline{0}$.
  Let $\|\underline{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ denote the Euclidean norm. Let $f(\underline{x}) = 1/\|\underline{x}\|$. Calculate the stochastic differential $\mathrm{d}f(\underline{B}_t)$ and show that the drift term vanishes.
- $M_t = 1/\|\underline{B}_t\|$ is not a martingale, since $\mathbb{E}(M_t) \to 0$ as $t \to \infty$ (with proof). Why doesn't this contradict the theory developed earlier?

- Let $\underline{B}_t = (B_1(t), B_2(t), B_3(t))$ denote 3-dimensional Brownian motion started from $\underline{B}_0 = \underline{x}_0 \in \mathbb{R}^d$, $\underline{x}_0 \neq \underline{0}$. Let $T_r = \inf\{ t : \|\underline{B}_t\| = r\}$. Let $0 < a < \|\underline{x}_0\| < b$.
  1. Find $\mathbb{P}(T_a < T_b)$.
  2. Show that $(\underline{B}_t)$ never hits the origin.
- Stochastic harmonic oscillator: let us consider the following system of SDE's:
$$\mathrm{d}X_t = -Y_t \,\mathrm{d}B_t, \qquad \mathrm{d}Y_t = X_t \,\mathrm{d}B_t, \qquad X_0 = 1, \qquad Y_0 = 0.$$
  Let $Z_t = X_t^2 + Y_t^2$. Find Z_t.

26. Lecture

- Bessel process: Let $(\underline{B}_t)$ denote d-dimensional Brownian motion. Show that if $Y_t = \|\underline{B}_t\|$, then (Y_t) is a weak solution of the SDE
$$\mathrm{d}Y_t = \frac{d-1}{2}\frac{1}{Y_t} + \mathrm{d}\widetilde{B}_t.$$
- Stochastic logarithm: given an Ito process (U_t) and X_0, find the Ito process (X_t) for which
$$U_t = 1 + \int_0^t U_s \,\mathrm{d}X_s, \qquad t \geq 0.$$
- Itô diffusion processes (i.e., Itô processes that are also Markov)
- Famous examples of Itô diffusion processes (which ones are time-homogenous?)
- Sometimes the solution of a linear SDE is not a Markov process
- Strong/weak solution of SDE
- Lipschitz condition implies existence and uniqueness of strong solution of SDE (statement of thm)
- Argue that geometric Brownian motion is a time-homogeneous Markov process and find its transition probability density function.

27. Lecture

- Solve $\mathrm{d}X_t = X_t^3 \mathrm{d}t + X_t^2 \mathrm{d}B_t$ with $X_0 = 1$ and argue why existence of solutions for all $t \in \mathbb{R}_+$ fails.
  Hint: The solution is of form $X_t = f(B_t)$ for some deterministic function f.
- Solve $\mathrm{d}X_t = 3X_t^{1/3}\mathrm{d}t3 + X_t^{2/3}\mathrm{d}B_t$ with $X_0 = 0$ and argue why uniqueness of solutions fails.
  Hint: The solution is of form $X_t = f(B_t)$ for some deterministic function f.
- General linear SDE: Given the Ito processes (X_t), (Y_t), find the Ito process (V_t) for which $\mathrm{d}V_t = V_t \mathrm{d}X_t + \mathrm{d}Y_t$ and $V_0 = v_0$.
- Squared Bessel process: given an Ito process that solves $\mathrm{d}Y_t = \frac{n-1}{2}\frac{1}{Y_t} + \mathrm{d}B_t$, let $X_t = Y_t^2$. Show that $\mathrm{d}X_t = n\mathrm{d}t + 2\sqrt{X_t}\mathrm{d}B_t$.
- Hitting probabilities for Bessel process: Let $0 < d \neq 2$, let $\mathrm{d}Y_t = \frac{d-1}{2}\frac{1}{Y_t} + \mathrm{d}B_t$, $Y_0 = y_0$. Let $0 < a < y_0 < b$. Let $T_y = \inf\{ t : Y_t = y\}$. Find $\mathbb{P}(T_a < T_b)$.

28. Lecture

- Continuous dependence of the solution of SDE on the initial condition (with proof)
- Grönwall's lemma (with proof)
- Idea of the proof of existence of a solution to SDE: Picard-Lindelöf iteration

29. Lecture

- Phase transition of Bessel process
- d-dimensional BM is transient if $d \geq 3$.
- Expected hitting time of level b by a d-dimensional Bessel process starting from $0 < y_0 < b$. $(d \geq 2)$.
- Classical logistic equation
- Strong solution of stochastic logistic equation

30. Lecture

- Hitting probabilities for general diffusion process
- An equivalent characterisation of the recurrence of a general diffusion process

31. Lecture (Skipped this in 2023)

- Existence of solutions of SDE (with proof)

32. Lecture

- An equivalent characterisation of the recurrence of a general diffusion process
- Finding the stationary distribution of an Itô diffusion process
- Does BM have a stationary distribution?
- Stationary distribution of O-U -process (including convergence to stationarity)
- Stationary distribution of the Cox-Ingersoll-Ross process
- Positive recurrence implies recurrence (using the equivalent characterizations)

33. Lecture

- Donsker's theorem, heuristic proof using infinitesimal drift/variance (Skipped this in 2023)
- Scaling limit of Ehrenfest chain is the O-U process (heu proof using infinitesimal drift/variance)
- Scaling limit of critical branching process with immigration (heu proof using infinitesimal drift/variance)

34. Lecture

- Tanaka's formula
- Tanaka's SDE has a unique weak solution but no strong solution (proof)

35. Lecture

- Infinitesimal generator of a diffusion process
- The analogy between the infinitesimal generators of diffusion processes and birth-death chains (Skipped this in 2023)
- Infinitesimal generator characterizes transition rules of diffusion process
- Infinitesimal generator of a n-dimensional diffusion process driven by d-dimensional BM

- How to find a 2-dimensional SDE such that the corresponding infinitesimal generator is

$$Af = f_{xx} + 3f_{xy} + \frac{5}{2}f_{yy} + f_x - f_y$$

- Dynkin's formula
- Infinitesimal generator of d-dimensional BM
- Diffusions solve elliptic SDEs: Helmholtz's equation
- Diffusions solve parabolic SDEs: Feynman-Kac formula

36. Lecture

- Measure change, Radon-Nikodym derivative
- If $X \sim \mathcal{N}(0,1)$ under $\mathbb{P}$ and $u \in \mathbb{R}$, find $\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = m(X)$ such that $u + X \sim \mathcal{N}(0,1)$ under $\mathbb{Q}$
- find the Radon-Nikodym derivative of the measure under which a random walk with independent standard normal increments plus deterministic drift term looks like a random walk with independent standard normal increments
- Girsanov's theorem (with proof)
- Cameron-Martin theorem
- Find the joint probability density function of $\max_{0\le t\le 1} \widetilde{B}_t$ and $\widetilde{B}_1$, where $(\widetilde{B}_t)$ is BM with constant drift

37. Lecture

- Black-Scholes model, what is the fair price of a European call option?
- Hedging, derivation of Black-Scholes PDE
- Solution of Black-Scholes PDE, Black-Scholes formula