Stoch. Anal. HW assignment 1. Due 2023 March 9 midnight

Note: Each of the 4 questions is worth 10 marks. Write the solutions of different exercises on different pages. Students of the BMETE95MM42 course can ignore one of these exercises.

1. Let (X, Y) denote a pair of continuous random variables with joint density function $f(x, y)$. Assume that $\mathbb{E}(|Y|) < +\infty$. Let $f_X(x) = \int_{-\infty}^{\infty} f(x, y)\,\mathrm{d}y$ and define

$$H(x) = \int_{-\infty}^{\infty} \frac{f(x, y)}{f_X(x)} y\,\mathrm{d}y.$$

Prove that the random variable $H(X)$ satisfies the abstract definition of $\mathbb{E}(\, Y \,|\, \sigma(X)\,)$.

Hint: You have to check that $Z := H(X)$ satisfies the following properties: $\mathbb{E}(|H(X)|) < +\infty$, $H(X)$ is $\sigma(X)$-measurable and that $\mathbb{E}(Y\mathbb{1}_A) = \mathbb{E}(H(X)\mathbb{1}_A)$ for any $A \in \sigma(X)$. You can use that $A \in \sigma(X)$ if and only if $A = X^{-1}(B)$ for some Borel set $B \in \mathcal{B}(\mathbb{R})$. Use the „law of the unconscious statistician" which states that if the probability density function (p.d.f.) of Z is $g(z)$, then $\mathbb{E}(\varphi(Z)) = \int_{-\infty}^{\infty} \varphi(z)g(z)\,\mathrm{d}z$ for any function $\varphi : \mathbb{R} \to \mathbb{R}$ for which the integral makes sense.

2. Let X and Y denote the outcomes of two i.i.d. die rolls. Let $Z := X + Y$. Find $\mathbb{E}(Z \,|\, X)$ and $\mathbb{E}(X \,|\, Z)$.

3. The moment generating function $M : \mathbb{R} \to (0, \infty]$ of a random variable X is defined by

$$M(\lambda) = M_X(\lambda) = \mathbb{E}(e^{\lambda X}).$$

If we assume that there exists $\lambda_0 > 0$ such that $M(\lambda_0) < \infty$ and $M(-\lambda_0) < \infty$ then it is known that

$$\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} M(\lambda) = \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \mathbb{E}(e^{\lambda X}) \overset{(*)}{=} \mathbb{E}\big(\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} e^{\lambda X}\big) = \mathbb{E}(X^n e^{\lambda X}) \quad \text{for any} \quad \lambda \in (-\lambda_0, \lambda_0).$$

(The only non-trivial equation is marked by $(*)$: if you want to see a proof of the fact that expectation and differentiation can be interchanged in this case, write an e-mail to me.)

Note that this implies $\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} M(\lambda)\big|_{\lambda=0} = \mathbb{E}(X^n)$, hence the name „moment generating function".

   (a) Let $X \sim \mathcal{N}(0, 1)$ (standard normal). Calculate $M_X(\lambda)$.
   Hint: Use the „law of the unconscious statistician". You should also use that any integral of form $\int_{-\infty}^{\infty} \exp(-ax^2 + bx + c)\,\mathrm{d}x$ can be calculated by manipulating the identity $\int_{-\infty}^{\infty} g(x)\,\mathrm{d}x = 1$, where $g(x)$ is the p.d.f. of some random variable $\mathcal{N}(\mu, \sigma^2)$.

   (b) Let $Y \sim \mathcal{N}(\mu, \sigma^2)$ (normal random variable with expectation μ and variance σ^2). Calculate $M_Y(\lambda)$.
   Hint: You can do this without calculating any further integrals: you just have to use part (a) of this exercise and the fact that Y has the same distribution as $\sigma X + \mu$.

   (c) Let $Z \sim \mathcal{N}(0, \sigma^2)$. Use the Taylor expansion of $M_Z(\lambda)$ about $\lambda = 0$ to calculate $\mathbb{E}(Z^n)$, $n \geq 0$.
   Hint: You will only have to use the Taylor expansion of the exponential function in a clever way.

   (d) Let $Z \sim \mathcal{N}(0, \sigma^2)$. Let $X = Z^2$. Calculate the variance of X using the results of part (c).

4. Let (X, Y) denote a pair of continuous random variables with joint density function

$$f(x, y) = c \exp\left(-\frac{5}{2}x^2 - 3xy - y^2\right).$$

   (a) Find c.
   Hint: We must have $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,\mathrm{d}x\mathrm{d}y = 1$. You should also use the fact that any integral of form $\int_{-\infty}^{\infty} \exp(-ax^2 + bx + c)\,\mathrm{d}x$ can be calculated by manipulating the identity $\int_{-\infty}^{\infty} g(x)\,\mathrm{d}x = 1$, where $g(x)$ is the p.d.f. of some random variable $\mathcal{N}(\mu, \sigma^2)$.

   (b) Find the density function $f_X(x)$ of X. *Hint:* $f_X(x) = \int_{-\infty}^{\infty} f(x, y)\,\mathrm{d}y$.

   (c) Find the covariance $\mathrm{Cov}(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$. *Hint:* Bivariate version of the „law of the unconscious statistician": $\mathbb{E}(\varphi(X, Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(x, y)f(x, y)\,\mathrm{d}x\mathrm{d}y$ for any $\varphi : \mathbb{R}^2 \to \mathbb{R}$.

   (d) Find $\mathbb{E}(\, Y \,|\, \sigma(X)\,)$. *Hint:* You can use that $\mathbb{E}(\, Y \,|\, \sigma(X)\,) = H(X)$, where $H(x)$ is given above.