Stoch. Anal. HW assignment 10. Due 2023 June 1, 11pm

Note: Each of the 4 questions is worth 10 marks.

1. *Kolmogorov's backward equation.* Let (X_t) denote the solution of the SDE

$$\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t.$$

(Let us assume that both $\mu : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ are Lipschitz-continuous.)

Let us also consider the solution $u(t,x)$ of the parabolic PDE

$$\partial_t u(t,x) = \mu(x)\partial_x u(t,x) + \frac{1}{2}\sigma^2(x)\partial_{xx}u(t,x), \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \tag{1}$$
$$u(0,x) = f(x), \qquad x \in \mathbb{R}. \tag{2}$$

(Let us assume that all functions involved are bounded and continuous.)

Show that for each $T \geq 0$, the following probabilistic interpretation of u holds:

$$u(T,x) = \mathbb{E}(f(X_T) \,|\, X_0 = x).$$

Hint: For any given T, consider $M_t := u(T-t, X_t)$, $0 \leq t \leq T$.

2. Use the result of the previous exercise to show that the solution of the PDE

$$\partial_t u(t,x) = \alpha x \partial_x u(t,x) + \frac{1}{2}\beta^2 x^2 \partial_{xx}u(t,x), \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}_+, \tag{3}$$
$$u(0,x) = f(x), \qquad x \in \mathbb{R} \tag{4}$$

is equal to

$$u(t,x) = \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} f\big(x \exp\{\beta y + (\alpha - \beta^2/2)t\}\big) e^{-\frac{y^2}{2t}} \,\mathrm{d}y, \qquad t, x > 0.$$

3. *Kolmogorov's forward equation.* Let (X_t) denote the solution of the SDE $\mathrm{d}X_t = \mu(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t$, where μ and σ are Lipschitz. The solution is a time-homogeneous Markov process, so let us consider its transition probability density function $p_t(x,y)$, $t \geq 0, x, y \in \mathbb{R}$ (see lecture 10). Show that for any $x \in \mathbb{R}$, the following PDE is satisfied:

$$\frac{\partial}{\partial t} p_t(x,y) = -\frac{\partial}{\partial y}(\mu(y)p_t(x,y)) + \frac{1}{2}\frac{\partial^2}{\partial y^2}(\sigma^2(y)p_t(x,y)), \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}.$$

Hint: Take an arbitrary compactly supported smooth test function $f : \mathbb{R} \to \mathbb{R}$. First observe that $\mathbb{E}(f(X_t) \,|\, X_0 = x)$ can be expressed using $p_t(x,y)$. Calculate the derivative $\partial_t \mathbb{E}(f(X_t) \,|\, X_0 = x)$ using Itô's formula and the tower rule, then use integration by parts. You may assume that all functions involved are sufficiently regular for your calculations to make sense without giving technical details.

4. *Poisson's equation with Dirichlet boundary condition.* Given a compact domain $\mathcal{D} \subseteq \mathbb{R}^d$ with a smooth boundary $\partial\mathcal{D}$, moreover $g : \mathcal{D} \to \mathbb{R}$ and $f : \partial\mathcal{D} \to \mathbb{R}$ (both bounded and continuous), we say that the function $u : \mathcal{D} \to \mathbb{R}$ is the solution of Poisson's equation with Dirichlet boundary condition if $\frac{1}{2}\Delta u(\underline{x}) = g(\underline{x})$ for all $\underline{x} \in \mathcal{D}^o$ (the interior of $\mathcal{D}$) and $u(\underline{x}) = f(\underline{x})$ for all $\underline{x} \in \partial\mathcal{D}$. Apply the optional stopping theorem to an appropriately chosen martingale to show that the solution u has the following probabilistic representation:

$$u(\underline{x}) = \mathbb{E}\left(f(\underline{B}_\tau) - \int_0^\tau g(\underline{B}_s)\,\mathrm{d}s \,\middle|\, \underline{B}_0 = \underline{x} \right), \qquad \underline{x} \in \mathcal{D},$$

where $(\underline{B}_t)$ is a d-dim. Brownian motion and $\tau = \min\{\, t : \underline{B}_t \in \partial\mathcal{D}\}$ is the exit time of $(\underline{B}_t)$ from $\mathcal{D}^o$.