**Stoch. Anal. HW assignment 2. Due 2023 March 16, 23.59**

*Note:* Each of the 4 questions is worth 10 marks. Write the solutions of different exercises on different pages. Students of the BMETE95MM42 course can ignore one of these exercises.

1. Let $S_n = \xi_1 + \cdots + \xi_n$, where $\xi_1, \xi_2, \ldots,$ are i.i.d. and $\mathbb{P}(\xi_k = 1) = \mathbb{P}(\xi_k = -1) = \frac{1}{2}$, $k \geq 1$. That is: $(S_n)$ is a one-dimensional simple, symmetric random walk. Let $(\mathcal{F}_n)_{n\geq 0}$ denote the natural filtration of $(S_n)$.

   (a) Find the discrete Doob-Meyer decomposition of the process $(e^{\lambda S_n})_{n\geq 1}$ where $\lambda \in \mathbb{R}$, i.e., write $e^{\lambda S_n} = A_n + M_n$, where $(A_n)$ is a predictable process and $(M_n)$ is a martingale with zero expectation. Explicitly state the formula for $A_n$.
   *Hint:* You may want to use the notation $\cosh(\lambda) = \frac{1}{2}\left(e^{\lambda} + e^{-\lambda}\right)$.

   (b) Write $M_n$ as the discrete stochastic integral $(H \cdot S)_n$ of a predictable process $(H_n)$ with respect to the martingale $(S_n)$. Explicitly state the formula for $H_n$.
   *Hint:* Observe that $S_k - S_{k-1}$ can only take two possible values. Notation: $\sinh(\lambda) = \frac{1}{2}\left(e^{\lambda} - e^{-\lambda}\right)$.

2. Let $\mathbb{F} = (\mathcal{F}_n)_{n=0}^{\infty}$ be a filtration and assume that $\mathcal{F}_0$ is the trivial sigma-field. Let $(N_n)_{n=1}^{\infty}$ be a square integrable martingale, i.e., assume that $\mathbb{E}(N_n^2) < \infty$ for each $n$.

   (a) Show that $\mathbb{E}(\Delta N_{i-1}^2 \,|\, \mathcal{F}_{i-1}) = \mathbb{E}(N_i^2 \,|\, \mathcal{F}_{i-1}) - N_{i-1}^2 = \mathbb{E}((\Delta N_{i-1})^2 \,|\, \mathcal{F}_{i-1})$.

   (b) Find the discrete Doob-Meyer decomposition of the process $(N_n^2)$, i.e., write $N_n^2 = A_n + M_n$, where $(A_n)$ is a predictable process and $(M_n)$ is a martingale with zero expectation. Write down explicitly the formula for $A_n$.

   (c) Prove the *Pythagorean theorem for square-integrable martingales:*

$$\mathbb{E}(N_n^2) = \mathbb{E}(N_1^2) + \sum_{j=1}^{n-1} \mathbb{E}((\Delta N_j)^2)$$

   *Hint:* It is enough to take the expectation of both sides of the equation $N_n^2 = A_n + M_n$.

3. Let $X_i$, $i = 1, 2, \ldots$ be i.i.d. with $\mathbb{P}(X_i = 1) = 1/3$ and $\mathbb{P}(X_i = -1) = 2/3$. Let $S_n := X_1 + X_2 + \cdots + X_n$. In other words, $(S_n)$ is a biased random walk with downward drift.

   (a) For what values of $\lambda \in (0, +\infty)$ will $M_n := \lambda^{S_n}$ be a martingale?
   *Hint:* First calculate $\mathbb{E}(\lambda^{X_n})$, then calculate $\mathbb{E}(M_n \,|\, \mathcal{F}_{n-1})$.

   (b) For $a \in \mathbb{Z}$ let $T_a = \min\{\, n : S_n = a \,\}$. Use the optional stopping theorem for the above martingale $M_n$ and the stopping time $\tau = T_{-a} \wedge T_b$ in order to calculate the probability $\mathbb{P}(T_{-a} > T_b)$ (i.e., the probability of the event that the walker hits $b$ before hitting $-a$) where $a, b \geq 0$.

   (c) Calculate $\lim_{a\to\infty} \mathbb{P}(T_{-a} > T_b)$ for any $b = 0, 1, 2, \ldots$.

   (d) What is the probability that the walker ever hits level $b$? I.e., what is $\mathbb{P}(T_b < \infty)$, $b = 0, 1, 2, \ldots$?

   (e) Denote by $\mathcal{M} = \max_{n\in\mathbb{N}} S_n$. Show that $\mathcal{M}$ has geometric distribution and find its parameter.
   *Hint:* Observe that $\mathcal{M} \geq b$ if and only if $T_b < \infty$, $b = 0, 1, 2, \ldots$

   (f) Show that $\mathbb{P}(T_{-100} < \infty) = 1$. *Hint:* Repeat the argument used in (c) and (d).

4. Let $X_i$, $i = 1, 2, \ldots$ be i.i.d. with $\mathbb{P}(X_i = 1) = \frac{2}{3}$, $\mathbb{P}(X_i = -1) = \frac{1}{3}$. Let $S_n := X_1 + X_2 + \cdots + X_n$.
   Thus $(S_n)$ is a biased random walk with an upward drift. Let $(\mathcal{F}_n)_{n\geq 0}$ denote the filtration generated by the process $(S_n)$.

   (a) Find the discrete Doob-Meyer decomposition of the process $(S_n)$, i.e., write $S_n = A_n + M_n$, where $(A_n)$ is a predictable process and $(M_n)$ is a martingale with zero expectation. Explicitly state the simplest possible formula for $A_n$.

   (b) Denote by $\tau = \min\{\, n : S_n = 100 \,\}$ the first time when the walker reaches level 100.
   Use part (a) and the optional stopping theorem to calculate $\mathbb{E}[\tau]$.
   *Instruction:* You don't have to fully verify that the optional stopping theorem can be applied here. In particular, don't prove that $\lim_{n\to\infty} \mathbb{E}(M_{n\wedge\tau}) = \mathbb{E}(M_\tau)$. But why do we have $\mathbb{P}(\tau < +\infty) = 1$?