Stoch. Anal. HW assignment 3. Due 2023 March 23 midnight

Note: Each of the 4 questions is worth 10 marks. Write the solutions of different exercises on different pages.

1. Let (B_t) and (B_t^*) denote independent standard Brownian motions.

   (a) *Time inversion:* Let $\widetilde{B}_t := t \cdot B_{1/t}$ if $t > 0$, $\widetilde{B}_0 := 0$. Show that $(\widetilde{B}_t)$ is a standard Brownian motion.

   (b) *Time reversal:* Let $X_t := B(1) - B(1-t)$ for any $0 \le t \le 1$. Show that $(X_t)_{0 \le t \le 1}$ is also a standard Brownian motion on the time interval $[0, 1]$.

   (c) *Superposition:* Let $\widehat{B}_t = \frac{1}{\sqrt{2}}(B_t + B_t^*)$. Show that $(\widehat{B}_t)$ is also a standard Brownian motion.

   Hint: We have already learnt two equivalent definitions of standard Brownian motion: in each of the sub-exercises (a), (b) and (c), use the definition that is most convenient.

2. Let (B_t) denote standard Brownian motion. Let $0 \le t_1 < t_2 < t_3$.

   (a) Write down the covariance matrix $\underline{\underline{C}}$ of the random vector $(B_{t_1}, B_{t_2}, B_{t_3})^T$.

   (b) Let Y be a standard normal r.v. independent from (B_t). Let $Z := \frac{t_3 - t_2}{t_3 - t_1} B_{t_1} + \frac{t_2 - t_1}{t_3 - t_1} B_{t_3} + aY$.
   In words: we obtain Z by linearly interpolating between B_{t_1} and B_{t_3} and then adding some independent noise aY. How to choose the constant $a > 0$ if we want $(B_{t_1}, Z, B_{t_3})^T$ to have the same joint distribution as $(B_{t_1}, B_{t_2}, B_{t_3})^T$?

   (c) What is the conditional density function of B_{t_2} given $B_{t_1} = x$ and $B_{t_3} = y$ for some $x, y \in \mathbb{R}$?

3. *Quadratic variation of Brownian motion.* Let (B_t) denote a standard Brownian motion. Given the partition $\Delta_n = \{t_0, t_1, \ldots, t_n\}$, $t_0 < t_1 < \cdots < t_n$, $t_0 = 0, t_n = t$, let $Q_n := \sum_{k=1}^n \big(B(t_k) - B(t_{k-1})\big)^2$.

   (a) Calculate $\mathbb{E}(Q_n)$ and $\mathrm{Var}(Q_n)$.
   Hint: You can find an important ingredient of this calculation in HW1.2(d)!

   (b) Calculate the quadratic variation of (B_t). More precisely, show that $[B]_t = t$.
   Hint: You have to show that $Q_n \xrightarrow{\mathbb{P}} t$ as $|\Delta_n| \to 0$. using part (a) and *Chebyshev's inequality.*

4. *The problem with stochastic integrals.* Recall the definition of the Stieltjes integral. The goal of this exercise is to „naively" try to calculate the integral

$$\int_0^t B(s)\, \mathrm{d}B(s),$$

   where (B_t) is standard Brownian motion. We will then soon realize that we get into trouble with our naive approach and that stochastic integrals behave differently from Stieltjes integrals.

   We consider a sequence of partitions (Δ_n) of $[0, t]$ satisfying $|\Delta_n| \to 0$. Let us define

$$L_n = \sum_{k=1}^n B(t_{k-1}) \cdot (B(t_k) - B(t_{k-1})), \qquad R_n = \sum_{k=1}^n B(t_k) \cdot (B(t_k) - B(t_{k-1})).$$

   In the case of L_n we chose the sample point t_k^* to be the left endpoint of the interval $[t_{k-1}, t_k]$, in the case of R_n we chose t_k^* to be the right endpoint. If this was a proper Stieltjes integral then we would have

$$\int_0^t B(s)\, \mathrm{d}B(s) = \lim_{|\Delta_n| \to 0} L_n = \lim_{|\Delta_n| \to 0} R_n.$$

   However, the truth turns out to be different:

   (a) Show that $R_n - L_n$ converges in probability as $|\Delta_n| \to 0$, but the limit is non-zero.

   (b) Find the limit of L_n (in probability) as $|\Delta_n| \to 0$.
   Hint: First calculate the limit of $L_n + R_n$ as $n \to \infty$.

   (c) For comparison, let us assume that $b : \mathbb{R}_+ \to \mathbb{R}$ is a continuously differentiable function satisfying $b(0) = 0$. Find the value of the Stieltjes integral $\int_0^t b(s)\, \mathrm{d}b(s)$.