Stoch. Anal. HW assignment 4. Due 2023 March 30, 11.00pm

Note: Each of the 4 questions is worth 10 marks. Write the solutions of different exercises on different pages.

1. *Laplace transform of the distribution of hitting time of Brownian motion with upward drift.*

   Let (B_t) denote the standard Brownian motion. Let $0 \le \mu < +\infty$. Let us define $X_t := B_t + \mu t$. We call (X_t) the Brownian motion with upward drift μ. Let $\tau = \min\{\, t \;:\; X_t = 1 \,\}$ the hitting time of level 1. You may assume without proof that $\mathbb{P}(\tau < +\infty) = 1$.

   (a) For any $\alpha \in \mathbb{R}$ find a constant $\beta \in \mathbb{R}$ such that (M_t) is a martingale, where $M_t := \exp(\alpha X_t - \beta t)$. *Hint:* For martingales related to Brownian motion, see Lecture 7.

   (b) Apply the optional stopping theorem to calculate $\mathbb{E}\left(e^{-\lambda \tau}\right)$ for any $\lambda \in \mathbb{R}_+$.
   Hint: You can use the following continuous-time form of the optional stopping theorem: if (M_t) is a martingale, τ is a stopping time satisfying $\mathbb{P}(\tau < +\infty) = 1$ and if there exists a constant $C \in \mathbb{R}_+$ such that $\mathbb{P}(|M_{t \wedge \tau}| \le C) = 1$ for any $t \ge 0$ then we have $\mathbb{E}(M_\tau) = \mathbb{E}(M_0)$.

2. Let $S_n = \xi_1 + \cdots + \xi_n$, where $\xi_1, \xi_2, \ldots,$ are i.i.d. and $\mathbb{P}(\xi_k = 1) = \mathbb{P}(\xi_k = -1) = \frac{1}{2}$, $k \ge 1$. Let $(\mathcal{F}_n)_{n \ge 0}$ denote the natural filtration of the simple random walk (S_n), i.e., $\mathcal{F}_n = \sigma(S_1, \ldots, S_n) = \sigma(\xi_1, \ldots, \xi_n)$.

   (a) Show that any martingale (M_n) with $M_0 = 0$ adapted to $(\mathcal{F}_n)_{n \ge 0}$ is a discrete stochastic integral $(H \cdot S)_n$ of a predictable process (H_n) with respect to the martingale (S_n). Explicitly state the formula for H_n. *Hint:* Since M_n is $\sigma(\xi_1, \ldots, \xi_n)$-measurable, there exists a function $\varphi_n : \{-1, 1\}^n \to \mathbb{R}$ such that $M_n = \varphi_n(\xi_1, \ldots, \xi_n)$. Please use this notation in your proof. Also note that an earlier homework was a special case of this exercise.

   (b) Show that if X is an $\mathcal{F}_{100}$-measurable random variable then $X = \mathbb{E}[X] + \sum_{k=1}^{100} Y_k \cdot (S_k - S_{k-1})$ for some random variables $Y_1, \ldots, Y_{100}$, where Y_k is $\mathcal{F}_{k-1}$-measurable, $k = 1, \ldots, 100$.
   Hint: How do you create a martingale $(M_n)_{n=0}^{100}$ out of the random variable X?

3. Recall the def. of stopping times (Lecture 4) and the σ-field $\mathcal{F}_\tau$ at the stopping time τ (Lecture 10).

   (a) Show that the collection of sets $\mathcal{F}_\tau$ is indeed a sigma-field. *Hint:* We defined σ-fields on Lecture 1.

   (b) Assume that we are in „discrete time” and define

   $$\mathcal{F}'_\tau = \{A \in \mathcal{F} \;:\; A \cap \{\tau = n\} \in \mathcal{F}_n,\ n \in \mathbb{N}\}.$$

   Show that $\mathcal{F}'_\tau = \mathcal{F}_\tau$.

   (c) Let (X_n) be a discrete-time stochastic process, let $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$ and let τ be a stopping time. Prove that X_τ is $\mathcal{F}_\tau$-measurable, i.e., that for any Borel set $B \in \mathcal{B}(\mathbb{R})$ we have $\{X_\tau \in B\} \in \mathcal{F}_\tau$.

   (d) Show that a discrete time Markov process (X_n) also has the strong Markov property.
   Hint: The plain Markov property is defined on Lecture 8 and the strong Markov property is defined on Lecture 10. You will have to show that $\mathbb{E}(f(X_{\tau+s}) \,|\, \sigma(X_\tau))$ satisfies the implicit definition (see Lecture 2) of $\mathbb{E}(f(X_{\tau+s}) \,|\, \mathcal{F}_\tau)$. The proof will involve an arbitrary event $A \in \mathcal{F}_\tau$, and here is a useful identity: $\mathbb{1}[A] = \sum_{n=0}^{\infty} \mathbb{1}[A \cap \{\tau = n\}]$.

4. Let $(B(t))$ denote the standard Brownian motion. Show that for any $x \ge 0$ we have

   $$\mathbb{P}\left(B(t) \ge x \,\middle|\, \min_{0 \le s \le t} B(s) \ge 0\right) = \exp\left(\frac{-x^2}{2t}\right).$$

   Hint: The probability of the condition is zero, so use the reflection principle (c.f. Lecture 10) to first show that

   $$\mathbb{P}(B(t) \ge x - \varepsilon \mid \min_{0 \le s \le t} B(s) \ge -\varepsilon) = \frac{\Phi\left(\frac{\varepsilon - x}{\sqrt{t}}\right) - \Phi\left(\frac{-\varepsilon - x}{\sqrt{t}}\right)}{2\Phi\left(\frac{\varepsilon}{\sqrt{t}}\right) - 1}, \qquad x > 0,\ \varepsilon > 0$$

   and then let ε go to 0.