Stoch. Anal. HW assignment 5. Due 2023 April 13, 11pm

Note: Each of the 5 questions is worth 10 marks. Write the solutions of different exercises on different pages.

1. We approximate the stochastic integral $\mathcal{Z} = \int_0^1 (2B(t)+1)\mathrm{d}B(t)$ with the sum

$$\mathcal{Z}_n = \sum_{k=1}^{n} \left(2B\left(\frac{k-1}{n}\right)+1\right)\cdot\left(B\left(\frac{k}{n}\right)-B\left(\frac{k-1}{n}\right)\right).$$

   (a) Write down the simple predictable function $(X_n(t))$, $0 \le t \le 1$ for which $\mathcal{Z}_n = \int_0^1 X_n(t)\mathrm{d}B(t)$.
   (b) What is the smallest value of n for which $\mathbb{E}\left[(\mathcal{Z}-\mathcal{Z}_n)^2\right] \le 10^{-3}$ holds?

2. *Discrete Itô formula.* We have learnt Itô's formula is class:

$$f(B_t) - f(B_0) = \int_0^t f'(B_s)\,\mathrm{d}B_s + \frac{1}{2}\int_0^t f''(B_s)\,\mathrm{d}s.$$

   The goal of this exercise is to find the discrete analogue of this formula.

   Let $S_n = \xi_1 + \cdots + \xi_n$, where $\xi_1, \xi_2, \ldots,$ are i.i.d. and $\mathbb{P}(\xi_k = 1) = \mathbb{P}(\xi_k = -1) = \frac{1}{2}$, $k \ge 1$. Let $(\mathcal{F}_n)_{n\ge 0}$ denote the natural filtration of the simple random walk (S_n), i.e., $\mathcal{F}_n = \sigma(S_1,\ldots,S_n) = \sigma(\xi_1,\ldots,\xi_n)$.

   Given some $f : \mathbb{Z} \to \mathbb{R}$ let us define

$$f^*(x) := \frac{f(x+1)-f(x-1)}{2}, \qquad f^{**}(x) := f(x+1) - 2f(x) + f(x-1)$$

   Note that $f^*(x)$ can be viewed as a discrete analogue of $f'(x)$, because the slope of the line that passes through the points $(x-1, f(x-1))$ and $(x+1, f(x+1))$ is $f^*(x)$.

   Note that $f^{**}(x)$ can be viewed as a discrete analogue of $f''(x)$ because the second derivative of the parabola that passes through the points $(x-1, f(x-1))$, $(x, f(x))$ and $(x+1, f(x+1))$ is $f^{**}(x)$.

   (a) Let us define $X_n = f(S_n)$. Find the discrete Doob-Meyer decomposition of the process (X_n), i.e., write $X_n = A_n + M_n$, where (A_n) is a predictable process and (M_n) is a martingale with zero expectation. Explicitly state the formula for A_n using the function $f^{**}(\cdot)$.
   (b) Write M_n as the discrete stochastic integral $(H \cdot S)_n$ of a predictable process (H_n) with respect to the martingale (S_n). Explicitly state the formula for H_n using the function $f^*(\cdot)$.
   (c) Put the results of (a) and (b) together to obtain a formula for $f(S_n) - f(S_0)$ that looks like the discrete version of the right-hand side of Itô's formula.

3. We say that two stochastic processes $(X(t))_{t\ge 0}$ and $(Y(t))_{t\ge 0}$ have the same law if for every choice of $n \ge 1$ and $0 \le t_1 < t_2 < \cdots < t_n$ the joint distributions of $(X(t_1), X(t_2), \ldots, X(t_n))$ and $(Y(t_1), Y(t_2), \ldots, Y(t_n))$ are the same. Denote by $(B(t))$ the standard Brownian motion.

   Given $\beta \in (-\frac{1}{2}, +\infty)$, find $\alpha \in \mathbb{R}_+$ and $c > 0$ so that $(X(t))_{t\ge 0}$ and $(Y(t))_{t\ge 0}$ have the same law, where

$$X(t) = B(c \cdot t^\alpha), \qquad Y(t) = \int_0^t s^\beta\,\mathrm{d}B(s).$$

   Briefly explain why $(X(t))_{t\ge 0}$ and $(Y(t))_{t\ge 0}$ have the same law using results seen in class.

   Hint: Gaussian processes.

4. *Equivalent definitions of the Brownian bridge.* Denote by $(B(t))$ the standard Brownian motion. Briefly argue that the three processes below (each one is defined for $0 \le t \le 1$) are Gaussian and that the three processes have the same law.

   (a) $B(t) - tB(1)$.
   (b) $(1-t)B(t/(1-t))$.
   (c) $(1-t)\int_0^t \frac{1}{1-s}\,\mathrm{d}B_s$

5. Give an explicit formula for the cumulative distribution function $F(x) = \mathbb{P}(X \le x)$, $x \in \mathbb{R}$ of the random variable X, where

$$X = \int_0^1 \frac{e^{B_u}\,\mathrm{d}u}{2(e^{B_u}+1)^2} + \int_0^1 \frac{\mathrm{d}B_u}{1+e^{-B_u}}.$$

   Hint: You may use $\Phi(\cdot)$, the c.d.f. of the standard normal distribution in your solution.