## Stoch. Anal. HW assignment 6. Due 2023 April 27, 11pm

*Note:* Each of the 5 questions is worth 10 marks. Write the solutions of different exercises on different pages.

1. *Multidimensional Brownian motion.* We say that $\big(\underline{B}(t)\big)$ is a $d$-dimensional Brownian motion if

$$\underline{B}(t) = \big(B_1(t), \dots, B_d(t)\big)^T,$$

where $(B_1(t)), \dots, (B_d(t))$ are independent one-dimensional standard Brownian motions. Thus $\big(\underline{B}(t)\big)$ is an $\mathbb{R}^d$-valued stochastic process, i.e., a random column vector that evolves in time.

Also, $\underline{B} : \mathbb{R}_+ \to \mathbb{R}^d$ can be viewed as a random vector-valued continuous function.

We say that the $d$-dimensional matrix $\underline{\underline{U}} = (U_{i,j})_{i,j=1}^d$ is orthogonal if $\underline{\underline{U}}^T \underline{\underline{U}} = \underline{\underline{U}}\underline{\underline{U}}^T = I$, where $I$ is the $d$-dimensional identity matrix. That is, the rows of $\underline{\underline{U}}$ form an orthonormal basis of $\mathbb{R}^d$. Also, the columns of $\underline{\underline{U}}$ form an orthonormal basis of $\mathbb{R}^d$. The linear transformation $\underline{x} \mapsto \underline{\underline{U}}\,\underline{x}$ is an isometry, i.e., the length of $\underline{\underline{U}}\,\underline{x}$ is equal to the length of $\underline{x}$. Typical examples of such transformations include reflections and rotations.

Show that if $\underline{\underline{U}}$ is a $d$-dimensional *orthogonal matrix* and if we define

$$\widetilde{\underline{B}}(t) = \underline{\underline{U}}\,\underline{B}(t), \quad t \in \mathbb{R}_+,$$

then $\Big(\widetilde{\underline{B}}(t)\Big)$ is also a $d$-dimensional Brownian motion.

*Hint:* You only need to check that $(\widetilde{B}_1(t)), \dots, (\widetilde{B}_d(t))$ are independent one-dimensional standard Brownian motions. You might want to use the equivalent definition of Brownian motion as a Gaussian process.

2. (a) Use Itō calculus to show that

$$M_2(t) = B_t^2 - t, \qquad M_4(t) = B_t^4 - 6tB_t^2 + 3t^2$$

are martingales. *Hint:* First calculate the stochastic differential of $(M_2(t))$ and $(M_4(t))$.

(b) Given $a \in \mathbb{R}_+$ let us define the stopping time

$$\tau = \min\{\, t \;:\; |B_t| = a \,\}.$$

Use the optional stopping theorem to calculate $\mathbb{E}(\tau)$ and $\mathbb{E}(\tau^2)$.

*Instruction:* You don't have to check that the optional stopping thm can be applied here – we will do that in class.

3. Find a simple explicit formula for a non-negative process $(X_t)$ satisfying

$$\mathrm{d}X_t = 4X_t \mathrm{d}B_t + 2X_t \mathrm{d}t, \qquad X_0 = 5.$$

*Hint:* First calculate the stochastic differential of $Y_t := \log(X_t)$ using Itô's formula for Itô processes and find a simple explicit formula expressing $Y_t$ in terms of $B_t$ and $t$.

4. Recall the definition of the Ornstein-Uhlenbeck process from lecture 13:

$$Y_t = e^{-\beta t} Y_0 + \sqrt{2\beta} \int_0^t e^{\beta \cdot (u-t)} \,\mathrm{d}B_u$$

(a) Calculate the stochastic differential of $Y_t$ and show that $(Y_t)$ is an Itô process by explicitly writing down the formula of the processes $(\mu_t)$ and $(\sigma_t)$ for which

$$Y_t = Y_0 + \int_0^t \mu_s \,\mathrm{d}s + \int_0^t \sigma_s \,\mathrm{d}B_s.$$

*Note:* The formulas for $\mu_s$ and $\sigma_s$ turn out to be rather simple. E.g., express $\mu_s$ in terms of $Y_s$.

(b) Calculate the quadratic variation $[Y]_t$ of the O.-U. process on the interval $[0, t]$.

5. *Discrete Tanaka's formula.*

(a) Let $x_0 \in \mathbb{Z}$. Write down the special case of the discrete Itô formula (see the solution of HW 5.2) when the function $f$ is chosen as $f(x) = |x - x_0|$. *Hint:* What is $f^*(x)$ and $f^{**}(x)$ in this case?

(b) Denote by $(S_n)$ the one dimensional simple symmetric random walk starting from $S_0 = 0$. Use (a) and the optional stopping theorem to calculate the expected number of steps that the walker spends at location $x_0 = 5$ before it reaches either level $-5$ or level 20.