## Stoch. Anal. HW assignment 6. Due 2023 April 2, 10.15pm

Note: Each of the 5 questions is worth 10 marks. Write the solutions of different exercises on different pages.

1. Multidimensional Brownian motion. We say that  $(\underline{B}(t))$  is a d-dimensional Brownian motion if

$$\underline{B}(t) = \left(B_1(t), \dots, B_d(t)\right)^T$$

where  $(B_1(t)), \ldots, (B_d(t))$  are independent one-dimensional standard Brownian motions. Thus  $(\underline{B}(t))$  is an  $\mathbb{R}^d$ -valued stochastic process, i.e., a random column vector that evolves in time.

Also,  $\underline{B}: \mathbb{R}_+ \to \mathbb{R}^d$  can be viewed as a random vector-valued continuous function.

We say that the *d*-dimensional matrix  $\underline{\underline{U}} = (U_{i,j})_{i,j=1}^d$  is orthogonal if  $\underline{\underline{U}}^T \underline{\underline{U}} = \underline{\underline{UU}}^T = I$ , where *I* is the *d*-dimensional identity matrix. That is, the rows of  $\underline{\underline{U}}$  form an orthonormal basis of  $\mathbb{R}^d$ . Also, the columns of  $\underline{\underline{U}}$  form an orthonormal basis of  $\mathbb{R}^d$ . The linear transformation  $\underline{x} \mapsto \underline{\underline{U}} \underline{x}$  is an isometry, i.e., the length of  $\underline{\underline{U}} \underline{x}$  is equal to the length of  $\underline{x}$ . Typical examples of such transformations include reflections and rotations.

Show that if  $\underline{U}$  is a *d*-dimensional *orthogonal matrix* and if we define

$$\underline{B}(t) = \underline{U} \underline{B}(t), \quad t \in \mathbb{R}_+,$$

then  $\left(\underline{\widetilde{B}}(t)\right)$  is also a *d*-dimensional Brownian motion.

*Hint:* You only need to check that  $(\tilde{B}_1(t)), \ldots, (\tilde{B}_d(t))$  are independent one-dimensional standard Brownian motions. You might want to use the equivalent definition of Brownian motion as a Gaussian process.

2. (a) Use It $\bar{o}$  calculus to show that

$$M_2(t) = B_t^2 - t, \qquad M_4(t) = B_t^4 - 6tB_t^2 + 3t^2$$

are martingales. *Hint:* First calculate the stochastic differential of  $(M_2(t))$  and  $(M_4(t))$ .

(b) Given  $a \in \mathbb{R}_+$  let us define the stopping time

$$\tau = \min\{t : |B_t| = a\}$$

Use the optional stopping theorem to calculate  $\mathbb{E}(\tau)$  and  $\mathbb{E}(\tau^2)$ . Instruction: You don't have to check that the optional stopping thm can be applied here – we will do that in class.

3. Find a simple explicit formula for a non-negative process  $(X_t)$  satisfying

$$\mathrm{d}X_t = 4X_t \mathrm{d}B_t + 2X_t \mathrm{d}t, \qquad X_0 = 5.$$

*Hint:* First calculate the stochastic differential of  $Y_t := \log(X_t)$  using Itô's formula for Itô processes and find a simple explicit formula expressing  $Y_t$  in terms of  $B_t$  and t.

4. Recall the definition of the Ornstein-Uhlenbeck process from lecture 13:

$$Y_t = e^{-\beta t} Y_0 + \sqrt{2\beta} \int_0^t e^{\beta \cdot (u-t)} \, \mathrm{d}B_u$$

(a) Calculate the stochastic differential of  $Y_t$  and show that  $(Y_t)$  is an Itô process by explicitly writing down the formula of the processes  $(\mu_t)$  and  $(\sigma_t)$  for which

$$Y_t = Y_0 + \int_0^t \mu_s \,\mathrm{d}s + \int_0^t \sigma_s \,\mathrm{d}B_s.$$

*Note:* The formulas for  $\mu_s$  and  $\sigma_s$  turn out to be rather simple. E.g., express  $\mu_s$  in terms of  $Y_s$ .

- (b) Calculate the quadratic variation  $[Y]_t$  of the O.-U. process on the interval [0, t].
- 5. Discrete Tanaka's formula.
  - (a) Let  $x_0 \in \mathbb{Z}$ . Write down the special case of the discrete Itô formula (see the solution of HW 5.2) when the function f is chosen as  $f(x) = |x x_0|$ . *Hint:* What is  $f^*(x)$  and  $f^{**}(x)$  in this case?
  - (b) Denote by  $(S_n)$  the one dimensional simple symmetric random walk starting from  $S_0 = 0$ . Use (a) and the optional stopping theorem to calculate the expected number of steps that the walker spends at location  $x_0 = 5$  before it reaches either level -5 or level 20.