Stoch. Anal. HW assignment 7. Due 2025 April 9, 10.15am

Note: Each of the 5 questions is worth 10 marks. Write the solutions of different exercises on different pages.

1. Hermite polynomials. We have already shown that the processes defined by

 $M_1(t) = B_t, \qquad M_2(t) = B_t^2 - t, \qquad M_3(t) = B_t^3 - 3tB_t, \qquad M_4(t) = B_t^4 - 6tB_t^2 + 3t^2$

are martingales.

- (a) Let us define $p_n(t,x) := \left. \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \exp(\lambda x \frac{\lambda^2}{2}t) \right|_{\lambda=0}$. Calculate $p_n(t,x)$ for n = 1, 2, 3, 4.
- (b) We say that the function u(t, x) is a solution of the reverse heat equation if $\frac{\partial}{\partial t}u(t, x) \equiv -\frac{1}{2}\frac{\partial^2}{\partial x^2}u(t, x)$. Show that for any value of the parameter $\lambda \in \mathbb{R}$, the function $\exp(\lambda x - \frac{\lambda^2}{2}t)$ solves the reverse heat equation and that the polynomials $p_n(t, x), n \in \mathbb{N}$ inherit this property.
- (c) Show that for any $n \ge 0$ the process defined by $M_n(t) := p_n(t, B_t)$ is a martingale.
- 2. (a) Show that $\widehat{M}_t = t^2 B_t 2 \int_0^t s B_s \, ds$ is a martingale by writing it as an Itō integral w.r.t. (B_t) .
 - (b) Show that if p(t, x) is a polynomial and

$$r(t,x) = \left(\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right)p(t,x)$$

then \widetilde{M}_t is a martingale, where $\widetilde{M}_t = p(t, B_t) - \int_0^t r(s, B_s) \, \mathrm{d}s$.

- (c) Find the Doob-Meyer decomposition $X_t = A_t + M_t$ of $X_t = (B_t^2 t)^2$, where A_t is an adapted process with finite total variation and M_t is a martingale. Write M_t as an Itō integral w.r.t. Brownian motion.
- 3. Given two independent Brownian motions $(B_1(t))$ and $(B_2(t))$ construct a third Brownian motion $(B_3(t))$ such that the mutual variation of $B_1(\cdot)$ and $B_3(\cdot)$ satisfies $[B_1, B_3]_t \equiv f(t)$, where $f : \mathbb{R}_+ \to \mathbb{R}$ is a given continuously differentiable function. What other conditions do we need about f for the exercise to be solvable?

Hint: Construct $(B_3(t))$ as $B_3(t) = \int_0^t g_1(u) dB_1(u) + \int_0^t g_2(u) dB_2(u) = Y_1(t) + Y_2(t)$ for some appropriately chosen $g_1, g_2 : \mathbb{R}_+ \to \mathbb{R}$. Use what we have learnt about the quadratic/mutual variation of Itô processes driven by *d*-dimensional Brownian motion (lecture 21). Show that $(B_3(t))$ is a Brownian motion using one of the equivalent characterizations of Brownian motion that we have learnt in class.

4. Log-optimal portfolio. You are trading shares at the stock market. The value of one stock at time t is

$$S_t = \sigma B_t + \mu t,$$

where (B_t) is standard Brownian motion and $\mu > 0$ (this is a toy example so let's not worry about the fact that S_t can become negative). At time t you hold C_t shares. Denote by $Y_0 > 0$ your initial wealth and by Y_t your total wealth at time t. It is OK to go in debt (i.e., it is OK if $C_t > Y_t$). Let us assume that you can't predict the future, so that (C_t) is left-continuous and adapted to (\mathcal{F}_t) , where $\mathcal{F}_t = \sigma(S_u, 0 \le u \le t)$. We have learnt in class (lecture 17) your net gain from trading shares at time t is

$$Y_t - Y_0 = \int_0^t C_u \, \mathrm{d}S_u$$

You trade in stocks until time T. Your goal is to maximize your expected rate of return $\mathbb{E}\left[\log\left(\frac{Y_T}{Y_0}\right)\right]$. What is the maximal expected rate of return achievable and the trading strategy that achieves it?

Hint: Let $X_t := \frac{C_t}{Y_t}$ denote the fraction of your wealth that you keep in shares (noting that $X_t > 1$ means that you are in debt). First calculate the stochastic differential $d\log(Y_t)$ and express the drift of $\log(Y_t)$ in terms of X_t . How to choose C_t if we want to maximize the drift? Switching back from differential form to integral form, you will be able to tell which trading strategy maximizes $\mathbb{E}\left[\log\left(\frac{Y_T}{Y_0}\right)\right]$.

5. Let $h : \mathbb{R}^2 \to \mathbb{R}$ be harmonic (i.e., $\Delta h \equiv 0$). Use Itō calculus, the optional stopping theorem and the rotation invariance of 2-dimensional Brownian motion (proved in Homework 6.1) to show

$$h(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} h(x_0 + r\cos(\varphi), y_0 + r\sin(\varphi)) \,\mathrm{d}\varphi$$