Stoch. Anal. HW assignment 7. Due 2023 May 4, 11pm

Note: Each of the 5 questions is worth 10 marks. Write the solutions of different exercises on different pages.

1. *Hermite polynomials.* We have already shown that the processes defined by

$$M_1(t) = B_t, \qquad M_2(t) = B_t^2 - t, \qquad M_3(t) = B_t^3 - 3tB_t, \qquad M_4(t) = B_t^4 - 6tB_t^2 + 3t^2$$

are martingales.

   (a) Let us define $p_n(t,x) := \frac{\mathrm{d}^n}{\mathrm{d}\lambda^n} \exp(\lambda x - \frac{\lambda^2}{2}t)\Big|_{\lambda=0}$. Calculate $p_n(t,x)$ for $n = 1,2,3,4$.

   (b) We say that the function $u(t,x)$ is a solution of the reverse heat equation if $\frac{\partial}{\partial t}u(t,x) \equiv -\frac{1}{2}\frac{\partial^2}{\partial x^2}u(t,x)$. Show that for any value of the parameter $\lambda \in \mathbb{R}$, the function $\exp(\lambda x - \frac{\lambda^2}{2}t)$ solves the reverse heat equation and that the polynomials $p_n(t,x), n \in \mathbb{N}$ inherit this property.

   (c) Show that for any $n \geq 0$ the process defined by $M_n(t) := p_n(t, B_t)$ is a martingale.

2. (a) Show that $\widehat{M_t} = t^2 B_t - 2\int_0^t sB_s \,\mathrm{d}s$ is a martingale by writing it as an Itō integral w.r.t. (B_t).

   (b) Show that if $p(t,x)$ is a polynomial and

$$r(t,x) = \left(\frac{\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right) p(t,x)$$

   then $\widetilde{M_t}$ is a martingale, where $\widetilde{M_t} = p(t,B_t) - \int_0^t r(s,B_s)\,\mathrm{d}s$.

   (c) Find the Doob-Meyer decomposition $X_t = A_t + M_t$ of $X_t = (B_t^2 - t)^2$, where A_t is an adapted process with finite total variation and M_t is a martingale. Write M_t as an Itō integral w.r.t. Brownian motion.

3. Given two independent Brownian motions $(B_1(t))$ and $(B_2(t))$ construct a third Brownian motion $(B_3(t))$ such that the mutual variation of $B_1(\cdot)$ and $B_3(\cdot)$ satisfies $[B_1, B_3]_t \equiv f(t)$, where $f : \mathbb{R}_+ \to \mathbb{R}$ is a given continuously differentiable function. What other conditions do we need about f for the exercise to be solvable?

   Hint: Construct $(B_3(t))$ as $B_3(t) = \int_0^t g_1(u)\mathrm{d}B_1(u) + \int_0^t g_2(u)\mathrm{d}B_2(u) = Y_1(t) + Y_2(t)$ for some appropriately chosen $g_1, g_2 : \mathbb{R}_+ \to \mathbb{R}$. Use what we have learnt about the quadratic/mutual variation of Itô processes driven by d-dimensional Brownian motion (lecture 21). Show that $(B_3(t))$ is a Brownian motion using one of the equivalent characterizations of Brownian motion that we have learnt in class.

4. *Log-optimal portfolio.* You are trading shares at the stock market. The value of one stock at time t is

$$S_t = \sigma B_t + \mu t,$$

where (B_t) is standard Brownian motion and $\mu > 0$ (this is a toy example so let's not worry about the fact that S_t can become negative). At time t you hold C_t shares. Denote by $Y_0 > 0$ your initial wealth and by Y_t your total wealth at time t. It is OK to go in debt (i.e., it is OK if $C_t > Y_t$). Let us assume that you can't predict the future, so that (C_t) is left-continuous and adapted to $(\mathcal{F}_t)$, where $\mathcal{F}_t = \sigma(S_u,\, 0 \leq u \leq t)$. We have learnt in class (lecture 17) your net gain from trading shares at time t is

$$Y_t - Y_0 = \int_0^t C_u \,\mathrm{d}S_u.$$

You trade in stocks until time T. Your goal is to maximize your expected rate of return $\mathbb{E}\left[\log\left(\frac{Y_T}{Y_0}\right)\right]$. What is the maximal expected rate of return achievable and the trading strategy that achieves it?

   Hint: Follow the solution strategy of HW3.C of the current *Markov Chains and Martinagles* course of Bálint Vető. First calculate $\mathrm{d}\log(Y_t)$.

5. Let $h : \mathbb{R}^2 \to \mathbb{R}$ be harmonic (i.e., $\Delta h \equiv 0$). Use Itō calculus, the optional stopping theorem and the rotation invariance of 2-dimensional Brownian motion (proved in Homework 6.1) to show

$$h(x_0, y_0) = \frac{1}{2\pi}\int_0^{2\pi} h\left(x_0 + r\cos(\varphi),\, y_0 + r\sin(\varphi)\right)\mathrm{d}\varphi$$