Stoch. Anal. HW assignment 8. Due 2023 may 11., 11pm

Note: Each of the 4 questions is worth 10 marks. Write the solutions of different exercises on different pages.

1. If $\underline{x} = (x, y) \in \mathbb{R}^2$, denote by $\|\underline{x}\| = \sqrt{x^2 + y^2}$ the Euclidean norm of $\underline{x}$. Let $\underline{B}_t = (B_1(t), B_2(t))$ denote a 2-dimensional Brownian motion starting from $\underline{x}_0 \neq \underline{0}$. Define $T_x = \inf\{\, t \,:\, \|\underline{B}_t\| = x \,\}$.

   The goal of this exercise is to show that

   - $(\underline{B}_t)$ always visits the ball of radius $a > 0$ around the origin but
   - $(\underline{B}_t)$ never hits the origin.

   (a) Calculate the stochastic differential $\mathrm{d}M_t$ of $M_t = \ln(\|\underline{B}_t\|)$.

   (b) Use the optional stopping theorem to calculate $\mathbb{P}[T_a < T_b]$ for any $0 < a < \|\underline{x}_0\| < b < +\infty$.

   (c) Show that $\mathbb{P}(T_a < +\infty) = 1$ for any $a > 0$. *Hint:* $b \to \infty$.

   (d) Show that $\mathbb{P}(T_0 < +\infty) = 0$. *Hint:* First $a \to 0$, then $b \to \infty$.

2. Let (X_t) be a Brownian motion with constant upward drift, i.e., the solution of the SDE

$$\mathrm{d}X_t = \mu \mathrm{d}t + \sigma \mathrm{d}B_t, \qquad X_0 = x_0 \in \mathbb{R}, \qquad \mu > 0,\ \sigma > 0.$$

   (a) Find a non-constant function $f : \mathbb{R} \to \mathbb{R}$ such that $f(X_t)$ is a martingale.

   (b) Use the above martingale and the optional stopping theorem to calculate $\mathbb{P}(T_a < T_b)$ where

$$a \leq x_0 \leq b, \qquad T_a = \min\{t \,:\, X_t = a\}, \qquad T_b = \min\{t \,:\, X_t = b\}$$

   (c) Find a function $g : \mathbb{R} \to \mathbb{R}$ such that $g(X_t) - t$ is a martingale and calculate $\mathbb{E}(\tau)$, where $\tau = T_a \wedge T_b$.

3. Solve the following stochastic differential equations:

   (a) Find (Z_t) such that $\mathrm{d}Z_t = B_t Z_t \mathrm{d}t + B_t Z_t \mathrm{d}B_t$ with initial condition $Z_0 = 1$.

   (b) Find (Y_t) such that $\mathrm{d}Y_t = Y_t \mathrm{d}t + B_t \mathrm{d}B_t$ with initial condition $Y_0 = 1$.

4. The O.-U. process is defined by the formula

$$X_t = X_0 e^{-\alpha t} + \sigma e^{-\alpha t} \int_0^t e^{\alpha s} \mathrm{d}B_s.$$

   (a) Manipulate the above formula to show that for any $s, t \in \mathbb{R}_+$ we have

$$X_{s+t} = X_s e^{-\alpha t} + \sigma e^{-\alpha t} \int_0^t e^{\alpha v} \mathrm{d}\widetilde{B}_v,$$

   where $\widetilde{B}(v) = B(s+v) - B(s)$ for any $v \geq 0$, thus $(\widetilde{B}_v)$ is also a standard Brownian motion which is independent from $\mathcal{F}_s$.

   (b) Show that (X_t) is a time-homogeneous Markov process (see lecture 10) and identify the corresponding transition density function $p_t(x, y)$.

   Hint1: It suffices to show $\mathbb{P}(X_{s+t} \in [y, y + \mathrm{d}y] \,|\, \mathcal{F}_s) = p_t(X_s, y)\mathrm{d}y$ in order to solve (b).

   Hint2: The O.-U. process is a Gaussian process, so if we fix t and x then $f(y) = p_t(x, y)$ will be the probability density function of a certain normal distribution.