Stoch. Anal. HW assignment 9. Due 2023 May 18, 11pm

Note: Each of the 4 questions is worth 10 marks. Write the solutions of different exercises on different pages.

1. Find (X_t) such that
$$\mathrm{d}X_t = X_t \mathrm{d}B_t + X_t \mathrm{d}t + B_t \mathrm{d}B_t, \qquad X_0 = 1.$$

2. Let $(u_t), (v_t)$ be left-continuous, adapted and almost surely bounded. Define
$$X_t := B_t + \int_0^t u_s \, \mathrm{d}s, \qquad M_t := \exp\left(\int_0^t v_s \, \mathrm{d}B_s - \frac{1}{2} \int_0^t v_s^2 \, \mathrm{d}s \right), \qquad Y_t := X_t M_t.$$
Given (u_t), how to choose (v_t) if you want both (M_t) and (Y_t) to be martingales?

3. We say that $(W_t)_{t \geq 0}$ is a Brownian motion *for* a filtration $(\mathcal{F}_t)_{t \geq 0}$ if
   (i) (W_t) is a standard Brownian motion,
   (ii) (W_t) is adapted to $(\mathcal{F}_t)$ and
   (iii) for any $0 \leq s \leq t$ the increment $W_t - W_s$ is independent of $\mathcal{F}_s$.

   Note that if (W_t) satisfies (i) and $(\mathcal{F}_t)$ is the natural filtration of (W_t) then (ii) and (iii) both hold. Also note that (i)&(ii)&(iii) together imply that (W_t) is a martingale with respect to $(\mathcal{F}_t)$. Throughout this course we tacitly assumed that our Brownian motions and filtrations satisfy (ii)&(iii). The goal of this exercise is to construct an example for which (i)&(ii) hold, but (iii) does not hold.

   Let (B_t) denote a standard Brownian motion and let $(\mathcal{F}_t)$ denote its natural filtration. Let us define
$$W_0 := 0, \qquad W_t := B_t - \int_0^t \frac{B_u}{u} \, \mathrm{d}u, \qquad t > 0. \tag{1}$$

   (a) Show that the above integral „makes sense" by showing that $\int_s^t \frac{B_u}{u} \, \mathrm{d}u$ converges in L_2 as $s \to 0_+$.
   Hint: Use integration by parts and Itō isometry (among other things).
   (b) Show that (W_t) is a standard Brownian motion.
   Hint: Observe that (W_t) is a Gaussian process...
   (c) Show that (W_t) satisfies (ii) but not (iii).
   Hint: (W_t) is an Itō process, but is it a martingale w.r.t. $(\mathcal{F}_t)$?

4. Let (X_t) be the solution of the SDE
$$\mathrm{d}X_t = \alpha X_t \, \mathrm{d}t + \sigma X_t^\beta \, \mathrm{d}B_t, \qquad X_0 = x_0,$$
where $\alpha, \beta, \sigma, x_0 \in \mathbb{R}$.
   (a) Compute $\mathbb{E}(X_t)$. *Hint:* Write the SDE in integral form and take the expectation.
   (b) Compute $\mathrm{Var}(X_t)$ when $\beta = 1/2$. *Hint:* Calculate $\mathrm{d}X_t^2$, rewrite in integral form, take expectation.