1 a) $\mathrm{Cov}(aB_3+bZ, B_3) = \mathrm{Cov}(aB_3, B_3) + 0 = 3a$

$\quad \parallel$

$\mathrm{Cov}(B_2, B_3) = 2$

$\Rightarrow a = \frac{2}{3}$

$\mathrm{Var}(aB_3 + bZ) = a^2\,\mathrm{Var}(B_3) + b^2\,\mathrm{Var}(Z) = \frac{4}{3} + b^2$

$\quad \parallel$

$\mathrm{Var}(B_2) = 2$

$\Rightarrow b^2 = \frac{2}{3}$

$b = \sqrt{\frac{2}{3}}$

b) $\mu_{B_2|B_3} = 0 + \frac{2}{3}(B_3 - 0) = \frac{2}{3}B_3$

$\sigma_{B_2|B_3} = 2 - \frac{2^2}{3} = \frac{2}{3}$

$f_{B_2|B_3}(x|y) = \frac{\sqrt{3}}{2\sqrt{\pi}} e^{-\frac{1}{2}\left(\frac{x-\frac{2}{3}y}{\sqrt{2/3}}\right)^2}$

c) $B_3 \sim B_2 + N(0,1) \Rightarrow f_{B_3|B_2}(x|y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-y)^2}$

2 a) $A_n = A_1 + \sum_{i=2}^{n} \Delta A_{i-1}$

$\Delta A_{i-1} = \mathbb{E}(\Delta(2S_{i-1}-1)^2 \mid \mathcal{F}_{i-1}) = \mathbb{E}((2S_i-1)^2 - (2S_{i-1}-1)^2 \mid \mathcal{F}_{i-1})$

$= \mathbb{E}(4(S_{i-1}+\xi_i)^2 - 4S_{i-1} - 4\xi_i \cancel{+1} - 4S_{i-1}^2 + 4S_{i-1} \cancel{-1} \mid \mathcal{F}_{i-1}) =$

$= \mathbb{E}(4\xi_i^2 \mid \mathcal{F}_{i-1}) = 4$

$A_1 = \mathbb{E}(2S_1+1)^2 = 4 + 0 + 1 = 5$

$A_n = 1 + 4n$

$M_n = 4S_n^2 - 4S_n + 1 - A_n = 4(S_n^2 - S_n - n)$

b) We want $M_n = \sum_{i=1}^{n} H_i (S_i - S_{i-1}) = \sum_{i=1}^{n} H_i \xi_i$ where H_i is $\mathcal{F}_{i-1}$-measurable

$$\Delta M_{n-1} = H_n \xi_n$$

$$\Delta M_{n-1} = 4(S_n^2 - S_n - n - S_{n-1}^2 + S_{n-1} + (n-1)) =$$

$$= 4(S_{n-1}^2 + 2S_{n-1}\xi_n + 1 - S_{n-1} - \xi_n - n - S_{n-1}^2 + S_{n-1} + (n-1)) =$$

$$= 4(2S_{n-1}\xi_n - \xi_n) = 4(2S_{n-1} - 1)\xi_n$$

Thus $H_i = 4(2S_{n-1} - 1)$

c) Pythagorean theorem for square-integrable martingales:

$$\mathbb{E}(M_n^2) = \mathbb{E}(M_1^2) + \sum_{i=1}^{n-1} \mathbb{E}((\Delta M_i)^2)$$

$$Var(X_n) = \mathbb{E}((X_n - \mathbb{E}(X_n))^2) = \mathbb{E}(M_n^2) \cancel{= 4\mathbb{E}((1-\xi_1-1)^2) +}$$

$$\cancel{+4\mathbb{E}} = \mathbb{E}(4^2(1-\xi_1-1)^2) + \cancel{4}\sum_{i=1}^{n-1} \mathbb{E}(4^2(2S_{i-1}-1)^2\xi_i^2) =$$

$$= 16 + 16\sum_{i=1}^{n-1} \mathbb{E}(4S_{i-1}^2 - 4S_{i-1} + 1) = 16\left(n + \sum_{i=1}^{n-1} 4\mathbb{E}(S_{i-1}^2)\right) =$$

$$= 16\left(n + 4\sum_{i=0}^{n-2} i\right) = 16(n + 2(n-2)(n-3))$$