MAKE-UP MIDTERM 2. 2025.05.23.

① a) $M_t = \mathbb{E}(\mathbb{X} \mid \mathcal{F}_t) =$

$$\exp\left(2\cdot\int_0^t s\,dB_s\right)\cdot \underbrace{\mathbb{E}\left(\exp\left(2\cdot\int_t^3 s\,dB_s\right) \Big| \mathcal{F}_t\right)}_{\text{INDEP.}}$$

$$= \mathbb{E}\left(\exp\left(2\cdot\int_t^3 s\,dB_s\right)\right) = e^{\frac{1}{2}\sigma^2\cdot 2^2}, \text{ WHERE}$$

$$\sigma^2 = \int_t^3 s^2\,ds = \frac{1}{3}\cdot(3^3 - t^3) = 9 - \frac{t^3}{3}$$

THUS $M_t = \exp\left(2\cdot\int_0^t s\,dB_s + 2\cdot\left(9 - \frac{t^3}{3}\right)\right)$

b) $M_t = \exp(Y_t)$, $Y_t := 2\cdot\int_0^t s\,dB_s + 2\cdot\left(9 - \frac{t^3}{3}\right)$

$$dM_t \overset{\text{ITŌ'S}}{\underset{\text{FORMULA}}{=}} \exp(Y_t)\,dY_t + \frac{1}{2}\exp(Y_t)\,d[Y]_t =$$

$$= M_t\cdot(2\cdot t\,dB_t - 2\cdot t^2\,dt) + \frac{1}{2}\cdot M_t\cdot(2t)^2\,dt$$

$= 2t\,M_t\,dB_t + 0\cdot dt$ (dt TERM VANISHES, INDEED MARTINGALE)

c) $\underset{\mathbb{X}}{\overset{\shortparallel}{M_3}} = \underset{\mathbb{E}(\mathbb{X})}{\overset{\shortparallel}{M_0}} + \int_0^3 dM_t = e^{18} + \int_0^3 2t\,M_t\,dB_t$

(2) $dX_t = 4\cdot X_t^{3/4}\,dB_t + 6\cdot X_t^{1/2}\,dt \qquad X_0 = 16$

$M_t = g(X_t)$, THUS $dM_t = g'(X_t)\,dX_t + \frac{1}{2}g''(X_t)\,d[X]_t =$

$= g'(X_t)\cdot\left(4\cdot X_t^{3/4}\,dB_t + 6\cdot X_t^{1/2}\,dt\right) + \frac{1}{2}g''(X_t)\cdot 16\cdot X_t^{3/2}\,dt$

$= \left(g'(X_t)\cdot 6\cdot X_t^{1/2} + \frac{1}{2}g''(X_t)\cdot 16\cdot X_t^{3/2}\right)dt + \dots$

THUS: WANT: $g'(x)\cdot 6\cdot x^{1/2} + 8\cdot g''(x)\cdot x^{3/2} \equiv 0$

LET $v(x) = g'(x)$. WANT: $-\frac{6}{x} = \frac{v'(x)}{v(x)}\cdot 8$, THUS

$-6\cdot\ln(x) = \ln(v(x))\cdot 8 + C$, THUS $v(x) = C\cdot x^{-3/4}$

THUS $g(x) = C_1\cdot x^{1/4} + C_2$ (CHOOSE: $C_1 = 1$, $C_2 = 0$)

$M_t = g(X_t) = X_t^{1/4}$ IS A MARTINGALE.

$$\underset{\displaystyle (16)^{1/4} = 2}{M_0} \overset{\text{O.S.T.}}{=\!=} \mathbb{E}(M_{\tau\wedge\tau^*}) = \underbrace{g(81)}_{3}\cdot\overbrace{\mathbb{P}(\tau^* < \tau)}^{p} + \underbrace{g(1)}_{1}\cdot(1-p)$$

$2 = 3\cdot p + 1 - p = 2p + 1$, THUS

$p = \mathbb{P}(\tau^* < \tau) = \frac{1}{2}$