① a) $\Delta M_n = M_{n+1} - M_n =$

$$(X_n + Y_{n+1})^3 - a\cdot(n+1)\cdot(X_n + Y_{n+1}) - X_n^3 + a\cdot n\cdot X_n =$$

$$= 3\cdot X_n^2\cdot Y_{n+1} + 3\cdot X_n\cdot Y_{n+1}^2 + Y_{n+1}^3 - a\cdot X_n - a\cdot(n+1)\cdot Y_{n+1}$$

$$\mathbb{E}(\Delta M_n \mid \mathcal{F}_n) =$$

$$3\cdot X_n^2\cdot 0 + 3\cdot X_n\cdot \frac{2}{3} + 0 - a\cdot X_n - a\cdot(n+1)\cdot 0 =$$

$$= (2-a)\cdot X_n \overset{\text{WANT}}{=\!=} 0, \text{ THUS } \boxed{a = 2}$$

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b) $M_n = X_n^3 - 2\cdot n\cdot X_n \qquad \mathcal{T} := \min\{T_0, T_R\}$

$$X_0^3 = \mathbb{E}(M_0) \overset{\text{O.S.T.}}{=\!=} \mathbb{E}(M_{\mathcal{T}}) = \mathbb{E}(X_{\mathcal{T}}^3 - 2\cdot\mathcal{T}\cdot X_{\mathcal{T}}) =$$

$$= \mathbb{E}(X_{\mathcal{T}}^3 - 2\cdot\mathcal{T}\cdot X_{\mathcal{T}} \mid T_R < T_0)\cdot\frac{X_0}{R} +$$

$$\mathbb{E}(X_{\mathcal{T}}^3 - 2\cdot\mathcal{T}\cdot X_{\mathcal{T}} \mid T_0 < T_R)\cdot\left(1 - \frac{X_0}{R}\right) =$$

$$= \mathbb{E}(R^3 - 2\cdot T_R\cdot R \mid T_R < T_0)\cdot\frac{X_0}{R} +$$

$$\mathbb{E}(0^3 - 2\cdot T_0\cdot 0 \mid T_0 < T_R)\cdot\left(1 - \frac{X_0}{R}\right) =$$

$$= R^3\cdot\frac{X_0}{R} - 2\cdot R\cdot\mathbb{E}(T_R \mid T_R < T_0)\cdot\frac{X_0}{R}, \text{ THUS}$$

$$\boxed{\mathbb{E}(T_R \mid T_R < T_0) = \frac{1}{2}\cdot(R^2 - X_0^2)}$$

(2) $X = X_1 + X_2$, $X_1 = \int_0^2 B_t \, dB_t$ $X_2 = \int_2^4 B_t \, dB_t$

$$\mathbb{E}(X^2 | \mathcal{F}_2) = \underbrace{\mathbb{E}(X_1^2 | \mathcal{F}_2)}_{A} + 2 \cdot \underbrace{\mathbb{E}(X_1 \cdot X_2 | \mathcal{F}_2)}_{B} + \underbrace{\mathbb{E}(X_2^2 | \mathcal{F}_2)}_{C}$$

$$A = X_1^2 = \left(\tfrac{1}{2}(B_2^2 - 2) \right)^2$$

$$B = X_1 \cdot \mathbb{E}(X_2 | \mathcal{F}_2) = X_1 \cdot 0 = 0$$

$$C \overset{\text{CONDITIONAL ITŌ ISOMETRY}}{=} \mathbb{E}\left(\int_2^4 B_t^2 \, dt \,\Big|\, \mathcal{F}_2 \right) =$$

$$\mathbb{E}\left(\int_2^4 (B_t^2 - t) \, dt \,\Big|\, \mathcal{F}_2 \right) + \int_2^4 t \, dt =$$

$$\overset{\text{CONDITIONAL FUBINI}}{=} \int_2^4 \mathbb{E}(B_t^2 - t | \mathcal{F}_2) \, dt + \tfrac{1}{2}(4^2 - 2^2) =$$

$$\overset{\text{MARTINGALE}}{=} \int_2^4 (B_2^2 - 2) \, dt + 6 = 2 \cdot B_2^2 - 2 \cdot 2 + 6 =$$

$$= 2 \cdot B_2^2 + 2$$