① $dY_t = -\frac{Y_t}{1+t}dt + \frac{60}{1+t}dB_t$ $\quad Y_0 = \frac{2}{3}$

THIS IS AN INHOMOGENEOUS LIN. S.D.E.

HOMOGENIZED S.D.E:

$$dX_t = -\frac{X_t}{1+t}dt \iff X_t' = -\frac{X_t}{1+t} \iff$$

$$\frac{X_t'}{X_t} = \frac{-1}{1+t} \Leftrightarrow \ln(X_t) = -\ln(1+t) + \tilde{C} \iff$$

$$\iff X_t = C \cdot \frac{1}{1+t}$$

METHOD OF VARIATION OF CONSTANTS:

$$Y_t = C_t \cdot \frac{1}{1+t} \qquad dY_t = \frac{1}{1+t}dC_t + C_t \cdot \frac{-1}{(1+t)^2}dt =$$

$$= \frac{1}{1+t}dC_t - \frac{Y_t}{1+t}dt \overset{\text{WANT}}{=\!=} \frac{60}{1+t}dB_t - \frac{Y_t}{1+t}dt$$

WANT: $dC_t = 60\,dB_t \Rightarrow C_t = C_0 + 60 \cdot B_t$

$$\frac{2}{3} = Y_0 = C_0 \cdot \frac{1}{1+0} = C_0 \Rightarrow C_t = \frac{2}{3} + 60 \cdot B_t \Rightarrow$$

$$Y_t = \frac{\frac{2}{3} + 60 \cdot B_t}{1+t} \Rightarrow Y_1 \sim N\left(\frac{1}{3}, (30)^2\right) \Rightarrow$$

$$\mathbb{P}(Y_1 \le x) = \Phi\left(\frac{x - 1/3}{30}\right)$$

(2) $dZ_t = 8\,dt + \sqrt{8Z_t}\,dB_t$, $Z_0 = 60$

$M_t = g(Z_t) - t$

$dM_t = g'(Z_t)\,dZ_t + \frac{1}{2} g''(Z_t)\,d[Z]_t - dt =$

$= g'(Z_t)\cdot(8\,dt + \sqrt{8Z_t}\,dB_t) + \frac{1}{2} g''(Z_t)\cdot 8Z_t\,dt - dt$

$= (g'(Z_t)\cdot 8 + 4\cdot g''(Z_t)\cdot Z_t - 1)\,dt + \boxed{\ldots}\,dB_t$

WANT: $g'(z)\cdot 8 + 4\cdot g''(z)\cdot z - 1 \equiv 0$

THIS IS AN INHOM. LIN. O.D.E.

FIRST SOLVE HOMOGENIZED LIN. O.D.E:

$h'(z)\cdot 8 + 4\cdot h''(z)\cdot z \equiv 0$ $\quad \boxed{f := h'}$

$f(z)\cdot 8 + 4\cdot f'(z)\cdot z \equiv 0$

$\boxed{\frac{f'(z)}{f(z)} = \frac{-2}{z}} \Rightarrow \boxed{f(z) = z^{-2}}$

BACK TO INHOM EQ: $\eta(z) := g'(z)$

$\eta(z)\cdot 8 + 4\cdot \eta'(z)\cdot z \equiv 1$ VARIATION OF CONSTANTS: $\boxed{\eta(z) = \frac{C(z)}{z^2}}$

$\frac{C(z)}{z^2}\cdot 8 + 4\cdot\left(\frac{C'(z)}{z^2} - \frac{2C(z)}{z^3}\right)\cdot z \equiv 1 \Rightarrow C'(z) = \frac{z}{4} \Rightarrow$

$C(z) = \frac{z^2}{8} \Rightarrow \eta(z) = \frac{1}{8} \Rightarrow g(z) = \frac{z}{8} \Rightarrow \boxed{M_t = \frac{Z_t}{8} - t}$

THUS BY OPTIONAL STOPPING THM:

$$\frac{60}{8} = \mathbb{E}(M_0) = \mathbb{E}(M_{T_{120}}) = \mathbb{E}\left(\frac{2T_{120}}{8} - T_{120}\right) =$$

$$= \mathbb{E}\left(\frac{120}{8} - T_{120}\right) = \frac{120}{8} - \mathbb{E}(T_{120})$$

$$\mathbb{E}(T_{120}) = \frac{60}{8} = \frac{15}{2}$$