Midterm Exam - October 13, 2016, Stochastic Analysis

Family name __________________________ **Given name** __________________________

Signature __________________________ **Neptun Code** __________________________

No calculators or electronic devices are allowed. One formula sheet with 15 formulas is allowed.

1. Let $S_n = \xi_1 + \cdots + \xi_n$, where $\xi_1, \xi_2, \ldots,$ are i.i.d. and $\mathbb{P}(\xi_k = 1) = \mathbb{P}(\xi_k = -1) = \frac{1}{2}$, $k \geq 1$. Let $(\mathcal{F}_n)_{n \geq 0}$ denote the natural filtration of the simple random walk (S_n), i.e., $\mathcal{F}_n = \sigma(S_1, \ldots, S_n) = \sigma(\xi_1, \ldots, \xi_n)$.

   Denote by $T_a = \min\{\, n \,:\, S_n = a \,\}$ the hitting time of level $a \in \mathbb{Z}$.

   (a) (2 marks) Calculate $\mathbb{P}[T_{10} < T_{-20}]$, i.e., the probability that the walker reaches level 10 before reaching level -20 using the optional stopping theorem.

   (b) (3 marks) Calculate $\mathbb{E}[T_{10} \wedge T_{-20}]$, i.e., the expected time spent until the walker reaches level 10 or level -20 using the optional stopping theorem.

   Instructions: You can use that (S_n) and $(S_n^2 - n)$ are martingales without proof. You don't have to check that the conditions of the optional stopping theorem apply here.

2. Let (B_t) denote standard Brownian motion.

   (a) (1 mark) Find the covariance matrix of (B_2, B_5).

   (b) (1 mark) Calculate the conditional expectation of B_5 given the sigma-field generated by B_2.

   (c) (3 marks) Calculate the conditional expectation of B_2 given the sigma-field generated by B_5.

3. We use the notation of the first exercise of this midterm.

   (a) (2 marks) Let $N_n = S_n^2 - n$. Write N_n as the discrete stochastic integral $(H \cdot S)_n$ of a predictable process (H_n) with respect to (S_n), i.e., find the value of H_n and show

$$N_n = \sum_{k=1}^{n} H_k \cdot (S_k - S_{k-1}).$$

   (b) (3 marks) Calculate $\mathbb{E}(N_n^2)$ using part (a). Find the simplest possible form of $\mathbb{E}(N_n^2)$.