First make-up midterm - December 7, 2016, 16.15-17.00, Stochastic Analysis

Family name ______________________________ **Given name** ______________________________

Signature ______________________________ **Neptun Code** ______________________________

No calculators or electronic devices are allowed. One formula sheet with 15 formulas is allowed.

1. Let X_i, $i = 1, 2, \ldots$ be i.i.d. with $\mathbb{P}(X_i = 1) = \frac{2}{3}$, $\mathbb{P}(X_i = -1) = \frac{1}{3}$. Let $S_n := X_1 + X_2 + \cdots + X_n$. Thus (S_n) is a biased random walk with an upward drift. Let $(\mathcal{F}_n)_{n \geq 0}$ denote the filtration generated by the process (S_n).
   (a) (3 marks) Find the discrete Doob-Meyer decomposition of the process (S_n), i.e., write $S_n = A_n + M_n$, where (A_n) is a predictable process and (M_n) is a martingale with zero expectation. Explicitly state the simplest possible formula for A_n.
   (b) (2 marks) Denote by $\tau = \min\{\, n : S_n = 100 \,\}$ the first time when the walker reaches level 100. Use part (a) and the optional stopping theorem to calculate $\mathbb{E}[\tau]$.
   Instruction: You don't have to verify that the optional stopping theorem can be applied here.

2. (5 marks) Let (B_t) denote standard Brownian motion. Let us define $\widetilde{B}_t = tB\left(\frac{1}{t}\right)$ for $t > 0$ and $\widetilde{B}_0 = 0$. Show that $(\widetilde{B}_t)$ is also a standard Brownian motion.
   Hint: Use the definition that starts like this: Brownian motion is a Gaussian process with ...

3. Let $S_n = \xi_1 + \cdots + \xi_n$, where $\xi_1, \xi_2, \ldots$, are i.i.d. and $\mathbb{P}(\xi_k = 1) = \mathbb{P}(\xi_k = -1) = \frac{1}{2}$, $k \geq 1$. Let $(\mathcal{F}_n)_{n \geq 0}$ denote the natural filtration of (S_n).
   (a) (2 marks) How to choose $C \in \mathbb{R}_+$ if we want $M_n = e^{S_n}/C^n$ to be a martingale?
   (b) (3 marks) Write $M_n - 1$ as the discrete stochastic integral $(H \cdot S)_n$ of a predictable process (H_n) with respect to the martingale (S_n). Explicitly state the formula for H_n.