First make-up midterm - December 7, 2016, 16.15-17.00, Stochastic Analysis

1. Let X_i, $i = 1, 2, \dots$ be i.i.d. with $\mathbb{P}(X_i = 1) = \frac{2}{3}$, $\mathbb{P}(X_i = -1) = \frac{1}{3}$. Let $S_n := X_1 + X_2 + \cdots + X_n$. Thus (S_n) is a biased random walk with an upward drift. Let $(\mathcal{F}_n)_{n\geq 0}$ denote the filtration generated by the process (S_n).
   (a) (3 marks) Find the discrete Doob-Meyer decomposition of the process (S_n), i.e., write $S_n = A_n + M_n$, where (A_n) is a predictable process and (M_n) is a martingale with zero expectation. Explicitly state the simplest possible formula for A_n.
   (b) (2 marks) Denote by $\tau = \min\{\, n : S_n = 100 \,\}$ the first time when the walker reaches level 100. Use part (a) and the optional stopping theorem to calculate $\mathbb{E}[\tau]$.
   Instruction: You don't have to verify that the optional stopping theorem can be applied here.

2. (5 marks) Let (B_t) denote standard Brownian motion. Let us define $\widetilde{B}_t = tB\left(\frac{1}{t}\right)$ for $t > 0$ and $\widetilde{B}_0 = 0$. Show that $(\widetilde{B}_t)$ is also a standard Brownian motion.
   Hint: Use the definition that starts like this: Brownian motion is a Gaussian process with ...

3. Let $S_n = \xi_1 + \cdots + \xi_n$, where $\xi_1, \xi_2, \dots,$ are i.i.d. and $\mathbb{P}(\xi_k = 1) = \mathbb{P}(\xi_k = -1) = \frac{1}{2}$, $k \geq 1$. Let $(\mathcal{F}_n)_{n\geq 0}$ denote the natural filtration of (S_n).
   (a) (2 marks) How to choose $C \in \mathbb{R}_+$ if we want $M_n = e^{S_n}/C^n$ to be a martingale?
   (b) (3 marks) Write $M_n - 1$ as the discrete stochastic integral $(H \cdot S)_n$ of a predictable process (H_n) with respect to the martingale (S_n). Explicitly state the formula for H_n.

Soltutions.

1. (a) $\Delta S_n = S_{n+1} - S_n = X_{n+1}$. $\Delta A_n = \mathbb{E}[\Delta S_n \,|\, \mathcal{F}_n] = \mathbb{E}[X_{n+1} \,|\, \mathcal{F}_n] = \mathbb{E}[X_{n+1}] = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$.
   $S_0 = 0$, thus $A_n = \sum_{i=0}^{n-1} \Delta A_i = \frac{n}{3}$ and $M_n = S_n - \frac{n}{3}$ is a martingale with $M_0 = 0$.
   (b) (Note that $\lim_{n\to\infty} S_n = +\infty$ by the law of large numbers, thus indeed $\mathbb{P}[\tau < \infty] = 1$.)
   $0 = M_0 = \mathbb{E}[M_0] = \mathbb{E}[M_\tau] = \mathbb{E}[S_\tau - \frac{\tau}{3}] = 100 - \mathbb{E}[\tau]/3$, thus $\mathbb{E}[\tau] = 300$.

2. $(\widetilde{B}_t)$ is a Gaussian process, because it arises as a deterministic function multiplied with a time-changed Brownian motion, where the time-change is also deterministic.
   We have $\mathbb{E}[\widetilde{B}_t] = t\mathbb{E}[B(\frac{1}{t})] = t \cdot 0 = 0$ and if $s < t$ then

$$\mathrm{Cov}[\widetilde{B}_s, \widetilde{B}_t] = \mathrm{Cov}[sB(\frac{1}{s}), tB(\frac{1}{t})] = st\mathrm{Cov}[B(\frac{1}{s}), B(\frac{1}{t})] = st \cdot (\frac{1}{s} \wedge \frac{1}{t}) = st\frac{1}{t} = s,$$

   therefore $(\widetilde{B}_t)$ is a Brownian motion.

3. (a) $\mathbb{E}[M_n \,|\, \mathcal{F}_{n-1}] = M_{n-1}\mathbb{E}[e^{\xi_n} \,|\, \mathcal{F}_{n-1}]/C = M_{n-1}\frac{e+e^{-1}}{2}\frac{1}{C}$, thus $C = \frac{e+e^{-1}}{2} = \mathrm{ch}(1)$.
   (b) $M_n - 1 = M_n - M_0 = \sum_{i=0}^{n-1} \Delta M_i$. We want to write $\Delta M_i = H_{i+1}\xi_{i+1}$, where H_{i+1} in $\mathcal{F}_i$-measurable.

$$\Delta M_n = M_{n+1} - M_n = M_n \cdot (e^{\xi_{n+1}}/\mathrm{ch}(1) - 1) = \frac{M_n}{\mathrm{ch}(1)}(e^{\xi_{n+1}} - \frac{e+e^{-1}}{2}) \overset{(*)}{=}$$
$$\frac{M_n}{\mathrm{ch}(1)}\frac{e-e^{-1}}{2}\xi_{n+1} = \frac{M_n}{\mathrm{ch}(1)}\mathrm{sh}(1)\xi_{n+1} = \mathrm{th}(1)M_n\xi_{n+1},$$

   where $(*)$ holds since ξ_{n+1} can only take two different values: ± 1. Thus $\Delta M_n = H_{n+1}\xi_{n+1}$, where $H_{n+1} = \mathrm{th}(1)M_n$. Thus $H_n = \mathrm{th}(1)M_{n-1} = \frac{e-e^{-1}}{e+e^{-1}}M_{n-1}$, indeed $\mathcal{F}_{n-1}$-measurable.