Midterm Exam - October 13, 2016, Stochastic Analysis

Family name ______________________ **Given name** ______________________

Signature ______________________ **Neptun Code** ______________________

No calculators or electronic devices are allowed. One formula sheet with 15 formulas is allowed.

1. Let $S_n = \xi_1 + \cdots + \xi_n$, where $\xi_1, \xi_2, \ldots,$ are i.i.d. and $\mathbb{P}(\xi_k = 1) = \mathbb{P}(\xi_k = -1) = \frac{1}{2}$, $k \geq 1$. Let $(\mathcal{F}_n)_{n\geq 0}$ denote the natural filtration of the simple random walk (S_n), i.e., $\mathcal{F}_n = \sigma(S_1, \ldots, S_n) = \sigma(\xi_1, \ldots, \xi_n)$.

   Denote by $T_a = \min\{\, n \, : \, S_n = a \,\}$ the hitting time of level $a \in \mathbb{Z}$.

   (a) (2 marks) Calculate $\mathbb{P}[T_{10} < T_{-20}]$, i.e., the probability that the walker reaches level 10 before reaching level -20 using the optional stopping theorem.

   (b) (3 marks) Calculate $\mathbb{E}[T_{10} \wedge T_{-20}]$, i.e., the expected time spent until the walker reaches level 10 or level -20 using the optional stopping theorem.
   Instructions: You can use that (S_n) and $(S_n^2 - n)$ are martingales without proof. You don't have to check that the conditions of the optional stopping theorem apply here.

2. Let (B_t) denote standard Brownian motion.

   (a) (1 mark) Find the covariance matrix of (B_2, B_5).

   (b) (1 mark) Calculate the conditional expectation of B_5 given the sigma-field generated by B_2.

   (c) (3 marks) Calculate the conditional expectation of B_2 given the sigma-field generated by B_5.

3. We use the notation of the first exercise of this midterm.

   (a) (2 marks) Let $N_n = S_n^2 - n$. Write N_n as the discrete stochastic integral $(H \cdot S)_n$ of a predictable process (H_n) with respect to (S_n), i.e., find the value of H_n and show

$$N_n = \sum_{k=1}^{n} H_k \cdot (S_k - S_{k-1}).$$

   (b) (3 marks) Calculate $\mathbb{E}(N_n^2)$ using part (a). Find the simplest possible form of $\mathbb{E}(N_n^2)$.

Solutions.

1. Let $\tau = T_{10} \wedge T_{-20}$

   (a) $0 = S_0 = \mathbb{E}(S_\tau) = 10 \cdot \mathbb{P}(T_{10} < T_{-20}) - 20 \cdot (1 - \mathbb{P}(T_{10} < T_{-20}))$, thus $\mathbb{P}(T_{10} < T_{-20}) = \frac{2}{3}$.

   (b) $0 = S_0^2 - 0 = \mathbb{E}(S_\tau^2 - \tau) = 100 \cdot \mathbb{P}(T_{10} < T_{-20}) + 400 \cdot \mathbb{P}(T_{10} > T_{-20}) - \mathbb{E}(\tau) = \frac{2}{3} \cdot 100 + \frac{1}{3} 400 - \mathbb{E}(\tau)$, thus $\mathbb{E}(\tau) = 200$.

2. (a) $\underline{\underline{C}} = \begin{pmatrix} 2 & 2 \\ 2 & 5 \end{pmatrix}$

   (b) $\mathbb{E}[B_5 \,|\, \sigma(B_2)] = \mathbb{E}[B_2 \,|\, \sigma(B_2)] + \mathbb{E}[(B_5 - B_2) \,|\, \sigma(B_2)] = B_2 + \mathbb{E}[B_5 - B_2] = B_2$.

   (c) Let $Z = B_2 - \lambda B_5$. Want: $\mathrm{Cov}(Z, B_5) = 0$. $\mathrm{Cov}(Z, B_5) = \mathrm{Cov}(B_2, B_5) - \lambda \mathrm{Cov}(B_5, B_5) = 2 - 5\lambda$. Thus $\lambda = \frac{2}{5}$. Now (Z, B_5) have multivariate normal distribution, so they are independent, thus $0 = \mathbb{E}[Z] = \mathbb{E}[Z \,|\, \sigma(B_5)] = \mathbb{E}[B_2 \,|\, \sigma(B_5)] - \frac{2}{5} B_5$, thus $\mathbb{E}[B_2 \,|\, \sigma(B_5)] = \frac{2}{5} B_5$.

3. (a) $\Delta N_k = (S_k^2 - k) - (S_{k-1}^2 - (k-1)) = (S_{k-1} + \xi_k)^2 - S_{k-1}^2 - 1 = 2S_{k-1}\xi_k$.
   Thus $H_k = 2S_{k-1}$ and $N_n = \sum_{k=1}^n \Delta N_k = \sum_{k=1}^n 2S_{k-1}\xi_k = \sum_{k=1}^n 2S_{k-1} \cdot (S_k - S_{k-1})$.

   (b) By the Pythagorean theorem for square-integrable martingales, we have

$$\mathbb{E}[N_n^2] = \sum_{k=1}^{n} \mathbb{E}[(\Delta N_k)^2] = \sum_{k=1}^{n} \mathbb{E}[4S_{k-1}^2 \xi_k^2] = 4\sum_{k=1}^{n} \mathbb{E}[S_{k-1}^2] = 4\sum_{k=1}^{n} (k-1) = 2(n-1)n$$