First make-up midterm - December 20, 2016, Stochastic Analysis

Family name ______________________________ **Given name** ______________________________

Signature ______________________________ **Neptun Code** ______________________________

No calculators or electronic devices are allowed. One formula sheet with 15 formulas is allowed.

1. Let X_i, $i = 1, 2, \ldots$ be i.i.d. with $\mathbb{P}(X_i = 1) = 1/2$ and $\mathbb{P}(X_i = -1) = 1/2$. Let $S_n := X_1 + X_2 + \cdots + X_n$. In other words, (S_n) is a simple symmetric random walk starting from $S_0 = 0$. Denote by $(\mathcal{F}_n)$ the natural filtration generated by (S_n).

   (a) (3 marks) Find the discrete Doob-Meyer decomposition of (S_n^2), i.e., write $S_n^2 = A_n + M_n$, where (A_n) is predictable and (M_n) is a martingale. Find the simplest possible form of A_n.

   (b) (2 marks) Write M_n as the discrete stochastic integral $(H \cdot S)_n$ of a predictable process (H_n) with respect to the martingale (S_n).

2. Recall from Paul Lévy's construction of Brownian motion that we defined $B^{(1)}(t) = tX + \frac{1}{2} f_1^0(t) Y_1^0$ for any $0 \le t \le 1$, where X and Y_1^0 are independent standard normal random variables and $f_1^0(t) = 1 - 2|x - \frac{1}{2}|$.

   (a) (2 marks) Calculate the covariance matrix of the pair of random variables $\left(B^{(1)}(1/4),\, B^{(1)}(3/4)\right)$.

   (b) (3 marks) Find the conditional expectation of $B^{(1)}(3/4)$ given the σ-field generated by $B^{(1)}(1/4)$.

3. Let (B_t) denote the standard Brownian motion. We define the hitting time $\tau = \min\{\, t \,:\, B_t = 2\,\}$.

   (a) (3 marks) Given $\beta \in \mathbb{R}$, how to choose $\alpha \in \mathbb{R}$ if we want $M_t = e^{\beta B_t - \alpha t}$ to be a martingale? Name the properties of conditional expectation and Brownian motion that you used in each step of your calculation.

   (b) (2 marks) Use the optional stopping theorem to calculate $\mathbb{E}\left(e^{-\lambda \tau}\right)$ for any $\lambda \in \mathbb{R}_+$.

   Instruction: You don't have to verify that the optional stopping theorem can be applied here.