First make-up midterm - December 13, 2016, 9.15-10.00, Stochastic Analysis

Family name ________________________ **Given name** ________________________

Signature ________________________ **Neptun Code** ________________________

No calculators or electronic devices are allowed. One formula sheet with 15 formulas is allowed.

1. Let $S_n = \xi_1 + \cdots + \xi_n$, where $\xi_1, \xi_2, \ldots,$ are i.i.d. and $\mathbb{P}(\xi_k = 1) = \mathbb{P}(\xi_k = -1) = \frac{1}{2}$, $k \geq 1$. Let $(\mathcal{F}_n)_{n\geq 0}$ denote the natural filtration of (S_n).
   (a) (3 marks) Find the discrete Doob-Meyer decomposition of the process $((3S_n - 2)^2)_{n\geq 1}$, i.e., write
   $$(3S_n - 2)^2 = A_n + M_n,$$
   where (A_n) is a predictable process and (M_n) is a martingale with zero expectation. Give a simple and explicit formula for A_n.
   (b) (2 marks) Write M_n as the discrete stochastic integral $(H \cdot S)_n$ of a predictable process (H_n) with respect to the martingale (S_n). Explicitly state the formula for H_n.

2. Let (B_t) denote the standard Brownian motion. Let $\sigma \in \mathbb{R}_+$ and define $\widetilde{B}_t = \sigma B_t$.
   Define $\tau = \min\{t \,:\, \widetilde{B}_t = 1\}$, the hitting time of 1 by $(\widetilde{B}_t)$.
   (a) (3 marks) Given $a \in \mathbb{R}$, how to choose $b \in \mathbb{R}$ if we want $M_t = e^{a\widetilde{B}_t - bt}$ to be a martingale?
   (b) (2 marks) Use the optional stopping theorem to calculate $\mathbb{E}\left(e^{-\lambda\tau}\right)$ for any $\lambda \in \mathbb{R}_+$.
   Instruction: You don't have to verify that the optional stopping theorem can be applied here.

3. (5 marks) Recall the definition of the stationary O.-U. process $X_t = e^{-\beta t} B(e^{2\beta t})$, where $\beta \in \mathbb{R}_+$.
   Calculate the conditional expectation of X_{-1} given the σ-algebra generated by X_1.