**First make-up midterm - December 15, 2016, Stochastic Analysis**

**Family name** ________________________ **Given name** ________________________

**Signature** ________________________ **Neptun Code** ________________________

**No calculators or electronic devices are allowed. One formula sheet with 15 formulas is allowed.**

1. Let $X_i$, $i = 1, 2, \dots$ be i.i.d. with $\mathbb{P}(X_i = 1) = 1/3$ and $\mathbb{P}(X_i = -1) = 2/3$. Let $S_n := 1 + X_1 + X_2 + \cdots + X_n$. In other words, $(S_n)$ is a biased random walk with downward drift starting from $S_0 = 1$.
   (a) (3 marks) For what values of $C$ will $M_n := C^{S_n}$ be a martingale?
   (b) (2 marks) For $a \in \mathbb{N}$ let $T_a = \min\{n : S_n = a\}$ denote the hitting time of $a$. Use the optional stopping theorem to calculate the probability $\mathbb{P}(T_5 < T_0)$, i.e., the probability that the walker reaches level 5 before reaching level 0.
   *Instruction:* You don't have to check that the conditions of the optional stopping theorem hold.

2. Let $(B_t)$ denote standard Brownian motion. Let us define

$$\mathcal{I}_n = \sum_{k=1}^{n} 2B\left(2\frac{k-1}{n}\right) \cdot \left(B\left(2\frac{k}{n}\right) - B\left(2\frac{k-1}{n}\right)\right), \quad \mathcal{Q}_n = \sum_{k=1}^{n} \left(B\left(2\frac{k}{n}\right) - B\left(2\frac{k-1}{n}\right)\right)^2$$

   (a) (3 marks) Show that both $\mathcal{I}_n$ and $\mathcal{Q}_n$ converge in probability to some random variables $\mathcal{I}$ and $\mathcal{Q}$ as $n \to \infty$ and find the simplest possible formula for the limits $\mathcal{I}$ and $\mathcal{Q}$.
   *Hint:* You can use without proof that the quadratic variation of Brownian motion is $[B]_t = t$.
   *Hint2:* Consider $\mathcal{I}_n + \mathcal{Q}_n$.
   (b) (2 marks) Use the Pyhtagorean theorem for square integrable martingales to calculate $\mathbb{E}[\mathcal{I}_n^2]$.

3. (5 marks) Let $(B_t)$ denote standard Brownian motion. Find the cumulative distribution function of

$$X = \min_{0 \le t \le 4} B_t,$$

   i.e., calculate $\mathbb{P}(X \le x)$ for all $x \in \mathbb{R}$. Please express your answer using the cumulative distribution function $\Phi(\cdot)$ of the standard normal random variable. Please explain how you used the *reflection principle* in your calculation by drawing a picture where the relevant quantities and random variables are clearly indicated.