Midterm Exam - April 20, 2023, Stochastic Analysis

Family name ______________________ **Given name** ______________________

Signature ______________________ **Neptun Code** ______________________

No calculators or electronic devices are allowed. One formula sheet with 15 formulas is OK.

1. (7 points) Let X_n denote the position of a one-dimensional simple symmetric random walker at time n. Let $(\mathcal{F}_n)_{n\geq 0}$ denote the natural filtration of (X_n).

   (a) Given $a_k \in \mathbb{R}_+, k \in \mathbb{N}$, how to choose $b_k \in \mathbb{R}_+$, $k \in \mathbb{N}$ if we want (M_n) to be a martingale, where

   $$M_n = \prod_{k=1}^{n} \frac{(a_k)^{X_k - X_{k-1}}}{b_k}, \qquad n = 0, 1, 2, \ldots$$

   (b) Write $M_n - 1$ as the discrete stochastic integral $(H \cdot X)_n$ of a predictable process (H_n) with respect to the martingale (X_n). Write the formula for H_n in a way that shows that (H_n) is indeed predictable.

2. (8 points) Let $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = t$. Let

   $$L_n = \sum_{k=1}^{n} e^{B(t_{k-1})} \cdot (B(t_k) - B(t_{k-1})), \qquad \mathcal{I} = \int_0^t e^{B_s}\, \mathrm{d}B_s.$$

   Show that

   $$\mathbb{E}((\mathcal{I} - L_n)^2) = \sum_{k=1}^{n} e^{2t_{k-1}} \cdot \left[\frac{1}{2}\left(e^{2(t_k - t_{k-1})} - 1\right) - 4\left(e^{\frac{1}{2}(t_k - t_{k-1})} - 1\right) + (t_k - t_{k-1})\right].$$

   Help: If $X \sim \mathcal{N}(0, \sigma^2)$ then the moment generating function of X is $M(\lambda) = e^{\frac{1}{2}\lambda^2\sigma^2}$.