Midterm Exam - March 26, 2025, Stochastic Analysis

Family name ________________________ **Given name** ________________________

Signature ________________________ **Neptun Code** ________________________

No calculators or electronic devices are allowed. One formula sheet with 15 formulas is OK.

1. (8 points) Let $X_1, X_2, \dots$ denote i.i.d. random variables with distribution

$$\mathbb{P}(X_k = 1) = \frac{2}{3}, \qquad \mathbb{P}(X_k = -2) = \frac{1}{3}. \tag{1}$$

Let $Y_n = X_1 + \cdots + X_n$. Let $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. Note that (Y_n) is a martingale. Let

$$M_n := \mathbb{E}\left[(Y_{100})^2 \,|\, \mathcal{F}_n\right], \quad 0 \le n \le 100.$$

(a) Calculate M_n.
Instruction: Your final answer should not contain (conditional) expectations.

(b) Show that the random variable $M_{100} - M_0$ can be written as a discrete stochastic integral of a predictable process (H_n) with respect to the martingale (Y_n), i.e., $M_{100} = M_0 + \sum_{k=1}^{100} H_k \cdot (Y_k - Y_{k-1})$ for some random variables $H_1, \dots, H_{100}$.
Instruction: Give a simple explicit formula for H_k from which it is clear that H_k is $\mathcal{F}_{k-1}$-measurable, $k = 1, \dots, 100$.

2. (7 points) Let (B_t) denote the standard Brownian motion. Let $\beta : [0,1] \to \mathbb{R}_+$ and let us define the stochastic processes $(X_t)_{0\le t\le 1}$ and $(Y_t)_{0\le t\le 1}$ by

$$X_t := \int_0^t \sqrt{2u}\,\mathrm{d}B_u - t^2 \int_0^1 \sqrt{2u}\,\mathrm{d}B_u, \qquad Y_t := (1-t^2)\int_0^t \beta(u)\,\mathrm{d}B_u.$$

How to choose the function $\beta : [0,1] \to \mathbb{R}_+$ if we want $(X_t)_{0\le t\le 1}$ and $(Y_t)_{0\le t\le 1}$ to have the same law?