Midterm Exam - March 26, 2025, Stochastic Analysis

1. (8 points) Let $X_1, X_2, \dots$ denote i.i.d. random variables with distribution $\mathbb{P}(X_k = 1) = \frac{2}{3}$ and $\mathbb{P}(X_k = -2) = \frac{1}{3}$. Let $Y_n = X_1 + \cdots + X_n$. Let $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. Note that (Y_n) is a martingale. Let $M_n := \mathbb{E}\left[(Y_{100})^2 \,|\, \mathcal{F}_n\right], \quad 0 \le n \le 100.$

   (a) Calculate M_n.

   (b) Show that the random variable $M_{100} - M_0$ can be written as a discrete stochastic integral of a predictable process (H_n) with respect to the martingale (Y_n), i.e., $M_{100} = M_0 + \sum_{k=1}^{100} H_k \cdot (Y_k - Y_{k-1})$ for some random variables $H_1, \dots, H_{100}$.

   Solution: Note that $\mathbb{E}(X_k) = 0$ and $\mathrm{Var}(X_k) = \mathbb{E}(X_k^2) = \frac{2}{3} \cdot 1^2 + \frac{1}{3} \cdot 2^2 = 2$.

   (a) $M_n = \mathbb{E}\left[(Y_n + (Y_{100} - Y_n))^2 \,|\, \mathcal{F}_n\right] = \mathbb{E}\left[Y_n^2 \,|\, \mathcal{F}_n\right] + 2\mathbb{E}\left[Y_n \cdot (Y_{100} - Y_n) \,|\, \mathcal{F}_n\right] +$ $\mathbb{E}\left[(Y_{100} - Y_n)^2 \,|\, \mathcal{F}_n\right] = Y_n^2 + 2Y_n \mathbb{E}\left[X_{n+1} + \cdots + X_{100}\right] + \mathbb{E}\left[(X_{n+1} + \cdots + X_{100})^2\right] =$ $Y_n^2 + 2Y_n \cdot 0 + \mathrm{Var}(X_{n+1} + \cdots + X_{100}) = Y_n^2 + (100 - n) \cdot 2$

   (b) Want: $M_{n+1} - M_n = H_{n+1} \cdot (Y_{n+1} - Y_n)$ for any $0 \le n \le 99$. Now we have $M_{n+1} - M_n = (Y_n + X_{n+1})^2 - Y_n^2 - 2 = 2Y_n X_{n+1} + X_{n+1}^2 - 2$. Noting that $Y_{n+1} - Y_n = X_{n+1}$, we want $H_{n+1} := \frac{2Y_n X_{n+1} + X_{n+1}^2 - 2}{X_{n+1}} = 2Y_n + X_{n+1} - \frac{2}{X_{n+1}}$ to be $\mathcal{F}_n$-measurable. And indeed: if $X_{n+1} = 1$ then $X_{n+1} - \frac{2}{X_{n+1}} = 1 - \frac{2}{1} = -1$ and if $X_{n+1} = -2$ then $X_{n+1} - \frac{2}{X_{n+1}} = -2 - \frac{2}{-2} = -1$. Thus

   $$H_{n+1} = 2Y_n - 1,$$

   which is indeed $\mathcal{F}_n$-measurable, thus $(H_n)_{n=1}^{100}$ is indeed a predictable process.

2. (7 points) Let (B_t) denote the standard Brownian motion. Let $\beta : [0,1] \to \mathbb{R}_+$ and let us define the stochastic processes $(X_t)_{0 \le t \le 1}$ and $(Y_t)_{0 \le t \le 1}$ by

   $$X_t := \int_0^t \sqrt{2u}\, \mathrm{d}B_u - t^2 \int_0^1 \sqrt{2u}\, \mathrm{d}B_u, \qquad Y_t := (1 - t^2) \int_0^t \beta(u)\, \mathrm{d}B_u.$$

   How to choose $\beta : [0,1] \to \mathbb{R}_+$ if we want $(X_t)_{0 \le t \le 1}$ and $(Y_t)_{0 \le t \le 1}$ to have the same law?

   Solution: Both $(X_t)_{0 \le t \le 1}$ and $(Y_t)_{0 \le t \le 1}$ are Gaussian processes, since they arise from Brownian motion (a Gaussian process) by linear operations. We have $\mathbb{E}[X_t] = \mathbb{E}[Y_t] = 0$ for all $0 \le t \le 1$, so the two processes will have the same law if they have the same autocovariance function. Let $0 \le s \le t \le 1$ thoughout.

   $$\mathrm{Cov}(X_s, X_t) = \int_0^s 2u\, \mathrm{d}u - t^2 \int_0^s 2u\, \mathrm{d}u - s^2 \int_0^t 2u\, \mathrm{d}u + s^2 t^2 \int_0^1 2u\, \mathrm{d}u =$$
   $$s^2 - t^2 s^2 - s^2 t^2 + s^2 t^2 = s^2 \cdot (1 - t^2).$$

   On the other hand, $\mathrm{Cov}(Y_s, Y_t) = (1 - s^2)(1 - t^2) \int_0^s \beta^2(u)\, \mathrm{d}u$. If we want this to be equal to $\mathrm{Cov}(X_s, X_t)$ then we want $\int_0^s \beta^2(u)\, \mathrm{d}u = \frac{s^2}{1 - s^2}$ for all $0 \le s \le 1$. The equality holds for $s = 0$, so differentiating both sides w.r.t. the variable s, we want $\beta^2(s) = \frac{2s}{(1 - s^2)^2}$, thus we have to choose

   $$\beta(s) := \frac{\sqrt{2s}}{1 - s^2}$$

   and then $(X_t)_{0 \le t \le 1}$ and $(Y_t)_{0 \le t \le 1}$ will have the same law.

   Remark: The processes $(X_t)_{0 \le t \le 1}$ and $(Y_t)_{0 \le t \le 1}$ both have the same law as $(\widetilde{B}(t^2))_{0 \le t \le 1}$, where $(\widetilde{B}(t))_{0 \le t \le 1}$ is a Brownian bridge (cf. HW5.4 and lecture 17).