**Midterm Exam - April 10, 2025, Stochastic Analysis**

**Family name** ____________________________ **Given name** ____________________________

**Signature** ____________________________ **Neptun Code** ____________________________

**No calculators or electronic devices are allowed. One formula sheet with 15 formulas is OK.**

1. Let $(B_t)$ denote a standard Brownian motion and let $(\mathcal{F}_t)$ denote its natural filtration. Let

$$Z_t = 1 + \int_0^t \frac{1}{\sqrt{1+s}} \, \mathrm{d}B_s.$$

   (a) (2 points) Calculate the probability of the event that the process $(Z_t)$ exits the interval $[0, 3]$ at its left endpoint (rather than at its right endpoint). *Instruction:* you may assume without proof that $(Z_t)$ almost surely does exit this interval.

   (b) (3 points) Find a function $\alpha : \mathbb{R}_+ \to \mathbb{R}$ with $\alpha(0) = 0$ such that $(M_t)$ is a martingale where $M_t := Z_t^2 - \alpha(t)$. *Instruction:* Please present a proof that does not use Itô's formula.

   (c) (3 points) Let $\tau$ denote the time that it takes for $(Z_t)$ to exit the interval $[0, 3]$. Calculate the expected value of $\alpha(\tau)$.

   *Instruction:* you can use the optional stopping theorem without checking its conditions.

2. Let $(B_t)$ denote a standard Brownian motion. Let

$$U := \int_0^2 B_t^3 \, \mathrm{d}B_t, \qquad V := \int_0^3 B_t \, \mathrm{d}B_t, \qquad Z := \int_0^1 e^{B_t} \, \mathrm{d}B_t.$$

   (a) (2 points) Calculate the variance of $U$.

   (b) (2 points) Calculate the covariance of $Z$ and $B_3$.

   (c) (3 points) Calculate the covariance of $V$ and $Z$.

   *Hint:* If $X \sim \mathcal{N}(0, \sigma^2)$ and $n = 2k$ then $\mathbb{E}[X^{2k}] = \frac{\sigma^{2k} \cdot (2k)!}{2^k k!}$ and $\mathbb{E}[e^{\lambda X}] = e^{\frac{1}{2}\lambda^2 \sigma^2}$.