STOCH. ANAL. MIDTERM 2025. IV. 10

FORMULAS THAT WE USE WHILE SOLVING THIS MIDTERM:

IF $M_t = \int_0^t X_s \, dB_s$ (WHERE (X_t) IS ADAPTED, ETC.), THEN (M_t) IS A MARTINGALE

$$\tilde{M}_t := \left(\int_0^t X_s \, dB_s \right)^2 - \int_0^t X_s^2 \, ds$$

$(\tilde{M}_t)$ IS ALSO A MARTINGALE

CONDITIONAL ITŌ ISOMETRY

$$\mathbb{E}\left(\int_s^t X_u \, dB_u \,\middle|\, \mathcal{F}_s \right) = 0$$

$$\mathbb{E}\left[\left(\int_s^t X_u \, dB_u \right)^2 \,\middle|\, \mathcal{F}_s \right) = \mathbb{E}\left(\int_s^t X_u^2 \, du \,\middle|\, \mathcal{F}_s \right)$$

$$\mathrm{Var}\left(\int_0^t X_s \, dB_s \right) = \mathbb{E}\left(\int_0^t X_s^2 \, ds \right)$$

$$\mathrm{Cov}\left(\int_0^t X_s \, dB_s , \int_0^t Y_s \, dB_s \right) = \mathbb{E}\left(\int_0^t X_s \cdot Y_s \, ds \right)$$

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① $\boxed{\mathcal{T} := T_0 \wedge T_3}$ (Z_t) IS A MARTINGALE

$p := \mathbb{P}(T_0 < T_3)$

a) $1 = Z_0 = \mathbb{E}(Z_0) \overset{\text{O.S.T.}}{=} \mathbb{E}(Z_{\mathcal{T}}) =$

$= 0 \cdot p + 3 \cdot (1-p)$ THUS $\boxed{p = 2/3}$

b) $Z_t^2 = 1 + 2 \cdot \underbrace{\int_0^t \frac{1}{\sqrt{1+s}} dB_s}_{} + \left(\int_0^t \frac{1}{\sqrt{1+s}} dB_s \right)^2$

THIS IS A MARTINGALE SO WE WANT $\alpha : \mathbb{R}_+ \to \mathbb{R}$ SUCH THAT $(\tilde{M}_t)$ IS A MARTINGALE, WHERE

$\boxed{\tilde{M}_t := (I_{0,t})^2 - \alpha(t)}$, WHERE $\boxed{I_{a,b} = \int_a^b \frac{1}{\sqrt{1+s}} dB_s}$

$s < t$: $\mathbb{E}(\tilde{M}_t \mid \mathcal{F}_s) =$

$\mathbb{E}\left((I_{0,s} + I_{s,t})^2 - \alpha(t) \mid \mathcal{F}_s \right) =$

$I_{0,s}^2 + 2 \cdot I_{0,s} \cdot \underbrace{\mathbb{E}(I_{s,t} \mid \mathcal{F}_s)}_{} + \underbrace{\mathbb{E}\left((I_{s,t})^2 \mid \mathcal{F}_s\right)}_{}$

$- \alpha(t) = I_{0,s}^2 + 0 + \mathbb{E}\left(\int_s^t \frac{1}{1+u} du \mid \mathcal{F}_s \right) -$

$- \alpha(t) =$

$$= \tilde{M}_s - (\alpha(t) - \alpha(s)) + \int_s^t \frac{1}{1+u}\,du = \tilde{M}_s$$

$\boxed{\text{WANT}}$

$$\text{THUS } \alpha(t) := \int_0^t \frac{1}{1+u}\,du = \ln(1+t)$$

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c) $1 = 1^2 - 0 = M_0 = \mathbb{E}(M_0) \overset{\text{O.S.T}}{=\!=} \mathbb{E}(M_J) =$

$\mathbb{E}(Z_J^2 - \alpha(J)) = p \cdot 0^2 + (1-p) \cdot 3^2 - \mathbb{E}(\alpha(J))$

$= 3 - \mathbb{E}(\alpha(J))$, THUS $\mathbb{E}(\alpha(J)) = 2$

$\boxed{p = \frac{2}{3} \mid \leftarrow \text{FROM PART a)}}$

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② a) $\mathrm{Var}(U) = \mathbb{E}(U^2) \overset{\text{ITŌ ISO.}}{=\!=} \mathbb{E}\left(\int_0^2 B_t^6\,dt\right) =$

$\overset{\text{FUBINI}}{=\!=} \int_0^2 \mathbb{E}(B_t^6)\,dt = \int_0^2 \frac{t^3 \cdot 6!}{2^3 \cdot 3!}\,dt =$

$= \frac{6 \cdot 5 \cdot 4}{8} \cdot \left[\frac{t^4}{4}\right]_0^2 = \frac{6 \cdot 5 \cdot 4}{8} \cdot 4 = 60$

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b) $B_3 = \int_0^3 1\,dB_s \quad \mathrm{Cov}\left(\int_0^3 1\,dB_s, \int_0^3 e^{B_s} \cdot \mathbb{1}[s \le 1]\,dB_s\right)$

$= \mathbb{E}\left(\int_0^1 1 \cdot e^{B_s}\,ds\right) = \int_0^1 \mathbb{E}(e^{B_s})\,ds = \int_0^1 e^{\frac{1}{2} \cdot 1^2 \cdot s}\,ds =$

$$= \int_0^1 e^{\frac{1}{2}s}\, ds = \left[2 \cdot e^{\frac{1}{2}s} \right]_0^1 = 2 \cdot \left(e^{1/2} - 1 \right)$$

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c) $$\mathrm{Cov}(V, Z) = \mathbb{E}\left(\int_0^1 B_t \cdot e^{B_t}\, dt \right) = ☺$$

IF $\mathbb{X} \sim N(0, \sigma^2)$ THEN

$$\mathbb{E}\left(e^{\lambda \cdot \mathbb{X}} \right) = e^{\frac{1}{2}\lambda^2 \cdot \sigma^2}, \text{ THUS}$$

$$\underbrace{\frac{d}{d\lambda} e^{\frac{1}{2}\lambda^2 \cdot \sigma^2}}_{\lambda \cdot \sigma^2 \cdot e^{\frac{1}{2}\lambda^2 \cdot \sigma^2}} = \mathbb{E}\left(\frac{d}{d\lambda} e^{\lambda \cdot \mathbb{X}} \right) = \underbrace{\mathbb{E}\left(\mathbb{X} \cdot e^{\lambda \cdot \mathbb{X}} \right)}$$

$$☺ = \int_0^1 \mathbb{E}\left(B_t \cdot e^{B_t} \right) dt = \int_0^1 1 \cdot t \cdot e^{\frac{1}{2}t}\, dt =$$

$$= \int_0^1 t \cdot e^{\frac{1}{2}t}\, dt \overset{\boxed{\text{INT. BY PARTS}}}{=} \left[t \cdot 2 \cdot e^{\frac{1}{2}t} \right]_0^1 - \int_0^1 1 \cdot 2 \cdot e^{\frac{1}{2}t}\, dt =$$

$$= 2 \cdot e^{1/2} - 2 \cdot \left[2 \cdot e^{\frac{1}{2}t} \right]_0^1 = 4 - 2 \cdot e^{1/2}$$